Income ranking, convergence speeds, and growth effects of inequality with two-dimensional adjustment

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Abstract

We present a growth model that converges to its balanced path by adjusting both inequality and the ratio of physical to human capital. This two-dimensional adjustment sheds light on some puzzling issues in models with one-dimensional adjustment. It shows in transition initial inequality hinders output growth but affects human capital growth ambiguously. It can produce the empirically estimated 2% rate of convergence for standard values of share parameters. It allows the convergence speed to vary according to initial states and education regimes, and explains why the OECD countries converge faster than others. It also permits rapid overtaking in income ranking among countries that share the same balanced path.

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JEL classification: E20; J24; O15; O41

1. Introduction

Convergence, capital accumulation, and income inequality have been the main themes in the literature on economic growth. This paper attempts to theoretically resolve some puzzling issues in regard to the speed of convergence, the effect of inequality on growth, and income ranking change among economies that have access to the same set of technology.

First, the empirically reported values of the annual rate of convergence vary significantly across countries or periods, and are substantially lower than those from model predictions. The former is in the range of 1-3% and higher in the OECD countries than in other countries, while the latter is of the order of 6-7% for standard values of share parameters in the production sector (see, e.g., Barro and Sala-i-Martin, 1992). Earlier effort reconciling this gap introduced externalities or learning-by-doing in production (e.g. Mankiw et al., 1992), but growth generated in this way is usually associated with an undesirable "scale effect" as pointed out by Matsuyama (1992). Incorporating adjustment costs of investment into a two-sector growth model pioneered by Uzawa (1964) and Lucas (1988), Ortigueira and Santos (1997) found that the convergence speed is a constant (rather than changing over time), depending on production parameters but unaffected by preference parameters.

Second, a negative effect of inequality on both output growth and capital accumulation was usually found in previous studies (e.g., Tamura, 1991, 1992; Glomm and Ravikumar, 1992; Perotti, 1993; Eckstein and Zilcha, 1994; Persson and Tabellini, 1994; Benabou, 1996), assuming either one type of capital or a constant ratio of one type of capital to another in a small open economy. One may wonder whether this result can still hold by taking into account the possibility that inequality could affect growth indirectly through its influence on the ratio of physical to human capital. As illustrated in Figure 1, there seems to be a non-negative simple correlation between inequality and the percentage increase in education attainment (a proxy for human capital) across countries, in contrast to the negative one between inequality and per capita GDP growth.[1]

[1]Due to the availability of data, this figure covers only 44 countries. The Gini coefficients during the period 1960-1970 are in Fields (1980), the growth rates of per capita income, averaged for 1970-1985, are based on Summer and Heston (1991), and the average schooling years of adult populations (1970-1985) are in Barro and Lee (1993).

Third, a typical drawback of the previous theoretical work is the lack of explanation to overtaking in income ranking among economies that share the same balanced path or steady-state income level. In the real world, income ranking changes from time to time, arising from miraculous growth in some countries along with stagnant development in some others (see e.g. Abramovitz, 1986; Lucas, 1993; Jones, 1997). But the existing models only permit partial catching up in income when a country starts with low inequality or with abundant labor or human capital.

These issues have important relevance. The speed of convergence tells how rapidly poor countries can narrow the income gap with their rich counterpart, whether convergence can be accelerated by policy, and whether policy should aim at the short or long horizon. Also, investigating how inequality may impinge differently on output growth, physical capital growth, and human capital growth helps set policy priority. Moreover, modeling income ranking change through growth miracles or stagnation, as was called for in Lucas (1993), may encourage poor nations to pursue active policy to combat poverty.

In this paper, we investigate these issues by developing a two-sector growth model with life-cycle savings, human capital, and inequality. The dynamic system comprises evolutions of both income distribution and the ratio of physical to human capital. This two-dimensional adjustment mechanism yields new insights into the puzzling issues that existed in the previous models with one-dimensional adjustment. We will show that inequality hinders income growth and physical capital growth but has an ambiguous (possibly positive) effect on human capital growth in transition. In particular, when high inequality initially decelerates human capital growth, the resulting relative abundance of physical capital leads to high wage rates but low interest rates, and thereby tends to accelerate human capital growth but decelerate physical capital growth.

We will also show that the convergence speed of output growth varies with initial states and over time, before reaching its stationary value that equals either the speed of inequality convergence or the speed of capital adjustment, whichever is lower. If inequality convergence sets the pace, then the convergence speed of output growth is determined by parameters in the education sector, and can be much lower than what was predicted by using standard parameters in the production sector.

In this case, public programs designed to mitigate inequality through spreading skills to the whole population (e.g. public schooling and public libraries) can accelerate the convergence of output growth. This may explain why the OECD countries that spend more on public education also converge faster than other countries. The previously estimated annual rate of convergence at 2% is obtained in this model for plausible parameterization.

Furthermore, rapid (stagnant) growth can occur in this model with two-dimensional adjustment. Intuitively, initial inequality and the initial ratio of physical to human capital, once all favorable (unfavorable) for growth, can reinforce each other for the extra push (drag) compared to models with only one-dimensional adjustment. The high (low) growth rate can remain above (below) normal for generations, due to the slow convergence of output growth, so income ranking can alter across nations that share the same balanced path.

The rest of the paper is organized as follows. Section 2 introduces the model. Sections 3 and 4 provide analytical and simulation results, respectively. Section 5 concludes.

2. The Model

This model has overlapping generations of agents who live for three periods. Young agents learn; middle-aged agents work, invest in child education, and save for old-age consumption; and old-aged agents consume their savings plus interest. Each middle-aged agent has one unit of labor time and devotes it to production, children's education, and leisure. Also, each middle-aged agent has one child, and each generation has a unit mass.

The preferences of a middle-aged agent are assumed as

$$V_t = \ln c_{1t} + \ln z_t + \eta \ln c_{2t+1} + \rho E_t V_{t+1}, \quad 0 < \eta < 1, \quad 0 < \rho < 1, \tag{1}$$

where c_{1t} stands for middle-age consumption, z_t leisure, c_{2t+1} old-age consumption, and $E_t V_{t+1}$ the expected welfare of his child. Here, η and ρ are subjective discount factors on utility components derived from old-age consumption and the welfare of the child, respectively. The use of a logarithmic utility function is based on two reasons. First, according to Ortigueira and Santos (1997), the rate of convergence is independent of the preference parameters of a CES utility function. As a special case of the CES function, the logarithmic specification should yield a comparable rate of convergence. Second, this log specification permits an analytical solution for the global characterization of the model with two-dimensional adjustment, particularly useful in cases that start outside the balanced path.

A middle-aged agent works for l_t units of time and earns $w_t l_t h_t$ where h_t is his human capital and w_t the wage rate per effective unit of labor. The middle-aged agent spends the earnings on middle-age consumption, the education of his child q_t, and savings s_t for old-age consumption:

$$c_{1t} = w_t l_t h_t - q_t - s_t, \tag{2}$$

$$c_{2t+1} = (1 + r_{t+1}) s_t. \tag{3}$$

The human capital of a child depends on his innate ability in learning (ξ), parental investments of income and time in his education, and the average human capital of the economy (H):

$$h_{t+1} = \xi q_t^{\varepsilon} [(1 - l_t - z_t) h_t]^{\beta(1-\varepsilon)} H_t^{(1-\varepsilon)(1-\beta)}, \quad 0 < \varepsilon < 1, \quad 0 < \beta < 1. \tag{4}$$

An individual's human capital fully depreciates at the end of his middle age. As in Benabou (1996), the distributions of ξ and h_t, $\phi(\xi)$ and $\psi(h_t)$, are all lognormal with $\ln \xi \sim \mathcal{N}(\mu_\xi, \Delta_\xi^2)$ and $\ln h_t \sim \mathcal{N}(\mu_t, \Delta_t^2)$, respectively. For expositional simplicity, let $\mu_\xi = -\Delta_\xi^2/2$ in the analysis (a different value of μ_ξ will be given later in simulations). The distribution of ability $\phi(\xi)$ is independent of the distribution of initial human capital $\psi(h_t)$. Also, investment in children's education is made before their ability shocks are revealed.

The production of the final good uses physical capital and effective labor as inputs according to

$$Y_t = D K_t^{\alpha} \left(\int_0^1 l_t h_t d\psi(h_t) \right)^{1-\alpha}, \quad 0 < \alpha < 1, \tag{5}$$

where K_t refers to aggregate physical capital that depreciates completely in one period, and $\int_0^1 l_t h_t d\psi(h_t)$ is aggregate effective labor. When one period corresponds to about 20 years, the implied annual rate of physical capital depreciation can be reasonably realistic. For ease of notation, let $k_t = K_t / \int_0^1 l_t h_t d\psi(h_t)$ be the ratio of physical capital to effective labor. Factors in production are compensated by their marginal products:

$$w_t = (1-\alpha) D k_t^{\alpha}, \tag{6}$$

$$1 + r_t = \alpha D k_t^{\alpha - 1}. \tag{7}$$

The final-good market equilibrium requires

$$K_{t+1} = \int_0^1 s_t dm(s_t), \tag{8}$$

where $m(s_t)$ is the distribution of s_t.

3. Equilibrium and Analytical Results

Starting with initial human capital h_t, a young parent solves the concave programming problem:

$$\begin{aligned} V(h_t, H_t, K_t, r_t, w_t) \;\; &= \max_{l_t, z_t, q_t, s_t} \Big\{ \ln(w_t l_t h_t - q_t - s_t) + \ln z_t + \eta \ln[(1 + r_{t+1}) s_t] \\ &\quad + \rho E_t V(h_{t+1}, H_{t+1}, K_{t+1}, r_{t+1}, w_{t+1}) \Big\}, \end{aligned} \tag{9}$$

subject to (4), taking the paths of H, K, r and w as given. The solution is given as follows (see the Appendix for derivation):

$$l_t = l = \frac{(1+\eta)[1 - \beta\rho(1-\varepsilon)]}{2 + \eta - \varepsilon\rho - \beta\rho(1-\varepsilon)}, \tag{10}$$

$$z_t = z = \frac{1 - \varepsilon\rho - \beta\rho(1-\varepsilon)}{2 + \eta - \varepsilon\rho - \beta\rho(1-\varepsilon)}, \tag{11}$$

$$q_t = \frac{\varepsilon\rho(1+\eta)}{2 + \eta - \varepsilon\rho - \beta\rho(1-\varepsilon)} w_t h_t \equiv \gamma_q w_t h_t, \tag{12}$$

$$s_t = \frac{\eta[1 - \varepsilon\rho - \beta\rho(1-\varepsilon)]}{2 + \eta - \varepsilon\rho - \beta\rho(1-\varepsilon)} w_t h_t \equiv \gamma_s w_t h_t, \tag{13}$$

where the time allocation (l, z) and the proportional output allocation (γ_q, γ_s) are all stationary.

This stationarity, implied by the log utility, allows us to proceed further to seek an analytical solution for the entire path of such a complicated system, as more general utility functions would restrict attention to asymptotical features around the balanced path through local approximation. From (8) and (13), physical capital accumulates according to

$$K_{t+1} = \gamma_s(1-\alpha)Dl^{-\alpha}K_t^{\alpha}H_t^{1-\alpha}. \tag{14}$$

So the growth rate of average physical capital, $g_{K,t} \equiv K_{t+1}/K_t - 1$, is determined by

$$\ln(1+g_{K,t}) = \Lambda + \Theta - (1-\alpha)\ln k_t, \tag{15}$$

where $\Lambda = \ln\gamma_s + \ln[(1-\alpha)D] - \ln l - \Theta$ and $\Theta = \varepsilon\ln[\gamma_q(1-\alpha)D] + \beta(1-\varepsilon)\ln(1-l-z)$. In this equation, there is a standard negative effect of the relative abundance of physical capital on physical capital growth, while inequality has no direct influence. However, it will soon become clear that physical capital growth is under an indirect influence of inequality through the ratio of physical capital to effective labor.

Similarly, from (4), (10), and (12), human capital accumulates in a family line according to

$$h_{t+1} = \xi[(1-\alpha)D\gamma_q]^{\varepsilon}(1-l-z)^{\beta(1-\varepsilon)}h_t^{\varepsilon+\beta(1-\varepsilon)}H_t^{(1-\varepsilon)(1-\beta)}k_t^{\varepsilon\alpha}. \tag{16}$$

Take logs on both sides of (16) and update the distribution of human capital through

$$\mu_{t+1} = \mu_\xi + \Theta + \varepsilon\alpha\ln k_t + \mu_t + (1-\varepsilon)(1-\beta)\Delta_t^2/2, \tag{17}$$

$$\Delta_{t+1}^2 = \Delta_\xi^2 + \Gamma^2\Delta_t^2, \tag{18}$$

where $\Gamma = \varepsilon + \beta(1-\varepsilon)$ and $0 < \Gamma < 1$ under $0 < \varepsilon < 1$ and $0 < \beta < 1$. The parameter Γ measures the importance of physical and time inputs in education, while $1-\Gamma$ measures the strength of the externality from average human capital. By (18), inequality measured by Δ_t^2 is convergent at a constant rate $1-\Gamma^2$ if $\Gamma < 1$, otherwise inequality would be ever increasing without limit. Average human capital growth, $g_{H,t} \equiv H_{t+1}/H_t - 1$, arises from (17), (18), $\mu_\xi = -\Delta_\xi^2/2$, and $H_t = \exp(\mu_t + \Delta_t^2/2)$:

$$\ln(1+g_{H,t}) = \Theta + \varepsilon\alpha\ln k_t - \Gamma(1-\Gamma)\Delta_t^2/2, \tag{19}$$

where human capital growth can be accelerated by relative abundance of physical capital but decelerated by inequality.

The growth rate of average output, $Y_{t+1}/Y_t - 1 \equiv g_{Y,t}$ follows (5), (8), (13), and (19):

$$\ln(1+g_{Y,t}) = \alpha\Lambda + \Theta - \alpha(1-\varepsilon)(1-\alpha)\ln k_t - \Gamma(1-\Gamma)(1-\alpha)\Delta_t^2/2, \tag{20}$$

where output growth depends negatively on both inequality and the ratio of physical capital to effective labor.

From (14) and (19), the ratio of physical capital to effective labor evolves according to

$$\ln k_{t+1} = \Lambda + \Gamma(1-\Gamma)\Delta_t^2/2 + \alpha(1-\varepsilon)\ln k_t. \tag{21}$$

An interesting feature in (21) is that inequality has a positive effect on the ratio of physical capital to effective labor. The reason is that inequality has a direct negative effect on human capital growth but no direct effect on physical capital growth. Note that, given the stationary labor supply, i.e. l in (10), the ratio of physical capital to effective labor, $K_t/(lH_t) = k_t$, is proportional to the ratio of physical to human capital K_t/H_t. In addition, the capital ratio k would converge at a constant rate $1-\alpha(1-\varepsilon)$ should inequality be ignored as in a standard two-sector growth model.

In (15) and (17)-(21), the dynamic system has been fully characterized by the two-dimensional evolution of Δ_t^2 and k_t. The remaining key step of the equilibrium analysis is to solve the entire path of the economy as a function of initial states, i.e. expressing (Δ_t^2, k_t) for all t as a function of (Δ_0^2, k_0). As to inequality adjustment, (18) implies

$$\Delta_t^2 = \Gamma^{2t}\Delta_0^2 + \frac{1-\Gamma^{2t}}{1-\Gamma^2}\Delta_\xi^2, \quad \Delta_\infty^2 = \frac{1}{1-\Gamma^2}\Delta_\xi^2. \tag{22}$$

As to capital adjustment, on the other hand, (21) and (22) produce

$$\ln k_t = \frac{1-\alpha^t(1-\varepsilon)^t}{1-\alpha(1-\varepsilon)}\left(\Lambda + \Gamma(1-\Gamma)\frac{\Delta_\infty^2}{2}\right)$$

$$+\frac{\Gamma(1-\Gamma)[\Gamma^{2t}-\alpha^t(1-\varepsilon)^t]}{2[\Gamma^2-\alpha(1-\varepsilon)]}(\Delta_0^2-\Delta_\infty^2) + \alpha^t(1-\varepsilon)^t\ln k_0, \tag{23}$$

$$\ln k_\infty = \frac{1}{1-\alpha(1-\varepsilon)}\left(\Lambda + \Gamma(1-\Gamma)\frac{\Delta_\infty^2}{2}\right). \tag{24}$$

To my best knowledge, this analytical expression of the whole dynamic path of the system with both the distribution of income and the accumulation of two-types of capital appears to be a novel achievement.

It is now ready to investigate what new insights this two-dimensional adjustment mechanism can offer. We first establish the tendency toward, and the properties of, the balanced path:

Proposition 1. *Starting from any initial state* (Δ_0^2, k_0)*, the economy is globally stable and converges to a unique balanced path consisting of* $(k_\infty, \Delta_\infty^2, g_\infty, r_\infty, w_\infty)$ *with* $g_{Y,\infty} = g_{H,\infty} = g_{K,\infty} \equiv g_\infty$*. The balanced growth rate increases with* γ_q *and* γ_s *but decreases with* Δ_∞^2*. Also,* k_∞ *and* w_∞ *all depend positively on* Δ_∞^2*, while* r_∞ *depends negatively on* Δ_∞^2*.*

(All proofs are relegated to the Appendix.) Like models with one-dimensional adjustment, the system with two-dimensional adjustment approaches its unique balanced path from any combination of initial inequality and initial capital stock. However, the characterization of the balanced path here is richer than in those previous models. In particular, the present model reveals the impact of long-run inequality on the long-run capital ratio, and hence on the wage and interest rates. The reason for this is as follows. Starting with H_t and K_t (or a given k_t), higher inequality Δ_t^2 with H_t being held constant, i.e. a mean-preserving spread in h_t, reduces average human capital in the next period H_{t+1} as a result of diminishing returns on individuals' human capital h_t under $\Gamma < 1$ in the education sector, by (19).[2] At the same time, this greater inequality Δ_t^2 has no effect on average physical capital in the next period K_{t+1} as was seen in (14), because from (8) and (13), K_{t+1} is proportional to the simple sum of workers' human capital. As a result, the ratio of physical to human capital in the next period, k_{t+1}, will rise.

How can initial states affect the transitional path? Examining this is equivalent to comparing two economies that share the same balanced path but start with different initial states.

[2] When there were constant returns on individuals' human capital (i.e. $\Gamma = 1$), children's human capital h_{t+1} would be proportional to, rather than a concave function of, their parents' h_t. So the mean-preserving rise in inequality in parents' generation Δ_t^2 would have no effect on the average human capital of children's generation H_{t+1}.

Proposition 2. *On the transitional path, (i)* k_0 *has positive effects on* $g_{H,t}$ *for* $t \geq 0$ *and on* (k_t, w_t) *for* $t > 0$*, but negative effects on* $(g_{K,t}, g_{Y,t})$ *for* $t \geq 0$ *and on* r_t *for* $t > 0$*; and (ii),* Δ_0^2 *has negative effects on* $(g_{K,t}, r_t)$ *for* $t > 0$*,* $g_{H,0}$ *and* $g_{Y,t}$ *for* $t \geq 0$*, positive effects on* (k_t, w_t) *for* $t > 0$*, but an ambiguous effect on* $g_{H,t}$ *for* $t > 0$*.*

In part (i), how the initial ratio of physical to human capital affects output growth and the accumulation of the two types of capital is standard, due to diminishing marginal products. Also standard are the negative effects of initial inequality in part (ii) on output growth from the initial period, physical capital growth after the initial period, and human capital growth within the initial period.

Unlike the standard view, however, the impact of initial inequality on human capital growth after the initial period can be positive, zero, or negative. The intuition comes from inequality's opposing effects on average human capital growth, directly through a concave education function on the one hand, and indirectly through the capital ratio with a one-period lag on the other hand. The positive effect of initial inequality on the ratio of physical to human capital leads to upward pressure on the wage rate. Through this equilibrium feedback, initial inequality creates upward pressure on human capital growth with a one-period lag, against its direct downward pressure that occurs immediately.[3] The net effect of initial inequality on average human capital growth is therefore negative immediately but ambiguous (possibly positive) afterwards in transition. For example, there is certainly a positive net effect of initial inequality Δ_0^2 on average human capital growth in the next period, $g_{H,1}$, provided that β is sufficiently small and that the production sector is more physical capital intensive than the education sector (i.e. $\alpha > \varepsilon$). Specifically, by (36), as β approaches 0, sign $\partial g_{H,1}/\partial \Delta_0^2 =$ sign $(\varepsilon\alpha - \Gamma^2) =$ sign $\varepsilon(\alpha - \varepsilon) > 0$ under $\alpha > \varepsilon$.

Moreover, one may doubt whether initial inequality has an ambiguous effect on the *level* of future average human capital as well, given initial H_0 and K_0 (or k_0). Examining this involves a

[3] This indirect effect would not arise if there were no physical input in the education sector (i.e. $\varepsilon = 0$) or no physical capital in the production sector (i.e. $\alpha = 0$) by (19), as in many previous models.

few steps. First, solving (17) gives:

$$\mu_j = j(\mu_\xi + \Theta) + \epsilon\alpha \sum_{t=0}^{j-1} \ln k_t + \mu_0 + \frac{(1-\epsilon)(1-\beta)}{2} \sum_{t=0}^{j-1} \Delta_t^2, \;\; j \geq 1. \tag{25}$$

From $\ln H_0 = \mu_0 + \Delta_0^2/2$, $H_j = \exp(\mu_j + \Delta_j^2/2)$, (18), and (25), we obtain:

$$\ln H_j = j(\mu_\xi + \Theta) + \epsilon\alpha \sum_{t=0}^{j-1} \ln k_t + \frac{(1-\epsilon)(1-\beta)}{2} \sum_{t=0}^{j-1} \Delta_t^2 + \frac{\Gamma^{2j}}{2}\Delta_0^2 - \frac{\Delta_0^2}{2} + \ln H_0, \;\; j \geq 1. \tag{26}$$

Equations (22), (23), and (26) imply:

$$\left(\frac{\partial H_j}{\partial \Delta_0^2}\right) \frac{2}{H_j} \;=\; \frac{\epsilon\alpha\Gamma(1-\Gamma)}{\Gamma^2 - \alpha(1-\epsilon)} \sum_{t=0}^{j-1} [\Gamma^{2t} - \alpha^t(1-\epsilon)^t] + (1-\epsilon)(1-\beta) \sum_{t=0}^{j-1} \Gamma^{2t} + \Gamma^{2j} - 1 \tag{27}$$

with

$$\sum_{t=0}^{j-1} [\Gamma^{2t} - \alpha^t(1-\epsilon)^t] = \frac{1-\Gamma^{2j}}{1-\Gamma^2} - \frac{1-\alpha^j(1-\epsilon)^j}{1-\alpha(1-\epsilon)}; \;\;\; \sum_{t=0}^{j-1} \Gamma^{2t} = \frac{1-\Gamma^{2j}}{1-\Gamma^2}; \;\;\; j \geq 1.$$

According to (27), $\partial H_1/\partial \Delta_0^2 < 0$, but the sign of $\partial H_j/\partial \Delta_0^2$ for $j \geq 2$ is ambiguous in general. That is, higher inequality in period 0 through a mean-preserving spread in individual human capital has a negative effect on the level of average human capital in period 1 but ambiguous effects afterwards. The intuition again comes from the opposing forces of initial inequality on human capital growth (i.e. direct vs. indirect effects). By contrast, without physical capital (i.e. $\alpha = 0$), the indirect force would not exist, so the net effect would always be negative: $\partial H_j/\partial \Delta_0^2$ under $\alpha = 0$ is signed by $(1-\epsilon)(1-\beta) - (1-\Gamma^2) = -\epsilon(1-\epsilon)(1-\beta) - (1-\epsilon)\beta[1-\beta(1-\epsilon)-\epsilon] < 0$.

Another important implication of this dynamic system is that different initial conditions may allow rapid or stagnant growth in a few periods such that income ranking among countries may alter.

Proposition 3. *Suppose that two economies start with initial states* (Δ_0^2, k_0, H_0) *and* $(\Delta_0'^2, k_0', H_0')$*, respectively. If* $(\Delta_0^2, k_0) < (\Delta_0'^2, k_0')$ *and* $H_0 = H_0'$*, then* $Y_0 < Y_0'$ *must hold, and* $Y_t > Y_t'$ *after a finite number of periods* t *provided that the inequality gap is sufficiently large relative to the gap in the physical to human capital ratio.*

The result here stands in stark contrast to that in the model with one-dimensional adjustment. Although a low physical to human capital ratio or low inequality alone may initially generate above-average growth for catching up in output per worker to certain extent in the latter type of models, this growth stimulus fades away monotonically when the capital ratio or inequality moves towards its steady-state level. Combining low inequality and relative abundance of human capital together could reinforce each other for rapid growth that allows overtaking in income comparison. Similarly, stagnation in development can arise when an economy starts with high inequality and relative lack of human capital. We will construct numerical examples later so as to see how fast the overtaking, or how persistent the stagnation, could be.

Finally, let us consider the speed of convergence. At the onset, it is useful to note that the speed of capital adjustment here is essentially the same as the speed of convergence derived from a neoclassical model of capital accumulation. Through a local approximation of the neoclassical growth model around its steady-state equilibrium path with a constant saving rate, the (annual) convergence speed equals $(1-\alpha)(n+x+\delta)$ where n is the rate of population growth, x the exogenous rate of technological progress, and δ the rate of depreciation. To be comparable, one can set $n=x=0$, and set $\delta=1$ under the assumption of complete depreciation of physical capital within one period in production. The resulting rate of convergence per period is equal to $1-\alpha$, the same as that in a special case of our model without physical inputs in education and without inequality (i.e. $\varepsilon=0$ and $\Delta_t^2=0$).

Another useful note is the connection between absolute and conditional convergence in the context of this model. In the literature on growth empirics, the convergence of per capita output has been found only conditional on controlling for other variables such as initial human capital. So let us start with the conditional convergence. Let $e_t=1-l_t-z_t$ be the time spent on education, and approximate $\ln H_t$ by be_{t-1} with $b>0$ since an indirect measure of human capital, i.e. education attainment, has been used in the growth empirics due to the lack of a direct measure. Incorporating this modification and substituting $\alpha\ln k_t=\ln Y_t-\ln(Dl)-\ln H_t$ into the output-growth equation (20), we obtain

$$\ln(1+g_{\mathrm{Y},t})=\tilde{\Theta}-(1-\epsilon)(1-\alpha)\ln Y_t+b(1-\epsilon)(1-\alpha)e_{t-1}-\Gamma(1-\Gamma)(1-\alpha)\Delta_t^2/2, \qquad (28)$$

where $\tilde{\Theta}$ is a constant. In (28), the growth rate of per capita output is related negatively to initial per capita output but positively to the controlled proxy for initial human capital, in line with the empirical evidence on the conditional convergence (e.g. Barro and Sala-i-Martin, 1992). According to (28), the speed of the conditional convergence of per capita output equals $(1-\epsilon)(1-\alpha)$, lower than the speed $1-\alpha$ in the neoclassical growth model or in a special case of this model without physical input in education. However, on the balanced path of this two-sector growth model there is generally no absolute convergence in income. The model can thus be consistent with both the conditional convergence and the absence of absolute convergence. The key point here is that richer countries on average have higher initial human capital so that per capita output and human capital are highly correlated and have offsetting effects on the subsequent growth rate of per capita output.

Differing from existing studies, convergence in this model involves two parts. On the one hand, income inequality converges to its steady-state solution at a rate $1-\Gamma^2 \equiv \lambda_\Delta$. On the other hand, the relative value of the two types of capital converges to its steady-state solution at a rate $1-\alpha(1-\varepsilon) \equiv \lambda_K$ if $\Delta_t = 0$ for all t. Denote $d_t = \ln(1+g_{Y,t}) - \ln(1+g_\infty)$ as the difference between the actual and balanced growth rates. Also, define the speed of the convergence of output growth as $\lambda_t \equiv |(|d_{t+1}| - |d_t|)/d_t|$, which measures the percentage change in the gap between initial and stationary growth rates.[4] If sign $d_{t+1} =$ sign d_t then $\lambda_t = |(d_{t+1}-d_t)/d_t|$, whereas if sign $d_{t+1} = -$ sign d_t then $\lambda_t = |(d_{t+1}+d_t)/d_t|$. Equations (20)-(35) imply

$$\lambda_t = \left| \frac{D_1\lambda_K\alpha^t(1-\varepsilon)^t\ln(k_0/k_\infty) + \{D_2\lambda_\Delta\Gamma^{2t} + D_3[\lambda_\Delta\Gamma^{2t} - \lambda_K\alpha^t(1-\varepsilon)^t]\}(\Delta_0^2-\Delta_\infty^2)}{-D_1\alpha^t(1-\varepsilon)^t\ln(k_0/k_\infty) - \{D_2\Gamma^{2t} + D_3[\Gamma^{2t} - \alpha^t(1-\varepsilon)^t]\}(\Delta_0^2-\Delta_\infty^2)} \right|$$

if sign $d_{t+1} =$ sign d_t;

$$\lambda_t = \left| \frac{D_1\lambda'_K\alpha^t(1-\varepsilon)^t\ln(k_0/k_\infty) + \{D_2\lambda'_\Delta\Gamma^{2t} + D_3[\lambda'_\Delta\Gamma^{2t} - \lambda'_K\alpha^t(1-\varepsilon)^t]\}(\Delta_0^2-\Delta_\infty^2)}{D_1\alpha^t(1-\varepsilon)^t\ln(k_0/k_\infty) + \{D_2\Gamma^{2t} + D_3[\Gamma^{2t} - \alpha^t(1-\varepsilon)^t]\}(\Delta_0^2-\Delta_\infty^2)} \right|$$

[4] In the neoclassical growth model, convergence usually refers to approaching a unique steady-state level of output per capita, and the speed of convergence indicates the responsiveness of per capita output growth to the gap between the initial state and the steady state. In the endogenous growth model of Ortigueira and Santos (1997), convergence refers to the system's returning to the balanced path and its speed is the same for all variables. In our model, however, output growth, inequality, and the capital ratio converge at different speeds λ, λ_Δ, and λ_K, respectively. We thus focus on the convergence of output growth to its stationary solution.

$$\text{if sign } d_{t+1} = -\text{sign } d_t, \tag{29}$$

where

$$\lambda'_K = 1 + \alpha(1-\varepsilon), \quad \lambda'_\Delta = 1 + \Gamma^2,$$

$$D_1 = \alpha(1-\alpha)(1-\varepsilon), \quad D_2 = \Gamma(1-\Gamma)(1-\alpha)/2, \quad D_3 = \frac{\alpha(1-\alpha)(1-\varepsilon)\Gamma(1-\Gamma)}{2[\Gamma^2 - \alpha(1-\varepsilon)]}.$$

According to (29), the speed of convergence λ_t in a transitional equilibrium changes over time t given initial states and the stationary solution. As t rises, we expect λ_t to converge to some stationary value λ_∞. We call λ_∞ the *stationary* speed of convergence.

Proposition 4. *The speed of the convergence of output growth, λ_t, varies with the gap between the initial state (k_0, Δ_0^2) and the steady state $(k_\infty, \Delta_\infty^2)$, and converges to the stationary rate $\lambda_\infty = \min\{\lambda_\Delta, \lambda_K\}$ with $\lambda_\Delta = 1 - \Gamma^2$ and $\lambda_K = 1 - \alpha(1-\varepsilon)$.*

One novel result in Proposition 4 is that the speed of the convergence of output growth varies with the gaps between initial and stationary inequality and between the initial and stationary ratios of physical to human capital. It can therefore explain the substantial variations in the convergence speed across countries and over time. This result would not arise, however, without inequality or without physical capital (i.e. $\alpha = 0$ or $\Delta_t = 0$). When $\alpha = 0$, for example, $\lambda_t = \lambda_\Delta$ for all t by (29), namely that, without physical capital in production, the convergence speed of output growth would be constant in all periods as in some previous models. When $\Delta_t = 0$ for all t, on the other hand, $\lambda_t = \lambda_K$ by (29); that is, with a homogeneous population, the convergence speed of output growth would be constant and equal to the speed of capital adjustment as was typically the case in the previous literature on convergence.

Another novel result in Proposition 4 is that the stationary speed of the convergence of output growth has its solution chosen from two distinctive values, whichever is lower. Intuitively, as output growth depends on adjustment in both inequality and the two types of capital, the slower one would drag the convergence of output growth. Specifically, when Γ^2 is greater than $\alpha(1-\varepsilon)$, the convergence of inequality is slow relative to that of the capital ratio, so the former (λ_Δ) governs the convergence speed of output growth. Similarly, when Γ^2 is smaller than $\alpha(1-\varepsilon)$, the convergence of inequality is rapid relative to that of the capital ratio, so the latter (λ_K) sets the pace for output-growth convergence.

Interestingly, since $\Gamma = \varepsilon + \beta(1-\varepsilon)$ represents the importance of private investment in education relative to the degree of the externality, differences in public education policy may lead to differences in the size of Γ and hence in the convergence speed of output growth. For example, free public schooling weakens the role of private investment in education relative to that of average human capital, and can thus accelerate the convergence of inequality as shown in Glomm and Ravikumar (1992) and Zhang (1996). Thus, when $\Gamma^2 > \alpha(1-\varepsilon)$, this accelerated inequality convergence leads to a higher stationary speed of the convergence of output growth. Put it differently, an economy with free public schooling has a higher stationary speed of the convergence of output growth than an economy with mainly private education, so long as private investment in education is sufficiently essential such that $\Gamma^2 > \alpha(1-\varepsilon)$. This result allows for variations in the speed of convergence among countries according to their different education regimes.

Indeed, in the real world the provision of public schooling varies substantially from country to country. In many developed countries (e.g. the OECD countries) there is universal access to free public schools at primary and secondary levels for more than 10 years of schooling, while in many developing countries such access is either unavailable or severely restricted. The same line of reasoning applies to the provision of public libraries, job training, and other public programs that aim at spreading knowledge and skills to the whole population for improvement in productivity. This argument receives support from the observed pattern of the speed of convergence across countries: the estimated speed of convergence is higher in the OECD countries than in the large sample of 98 countries according to Barro and Sala-i-Martin (1992) and Mankiw, Romer, and Weil (1992).

It is also interesting to note that the stationary speed of the convergence of output growth λ_∞ is determined by share parameters in the production sector and the education sector, and is independent of preference parameters. Previous studies have focused on the negative effect of increasing physical capital's share parameter in the production sector on the speed of convergence. The values of the speed of convergence in their calibrations, for the standard values of physical capital's share parameter in the production sector, are much higher than those in empirical studies.

In this model, if the pace of output-growth convergence is set by inequality adjustment, i.e. $\lambda_\infty = \lambda_\Delta = 1 - \Gamma^2 = 1 - [\varepsilon + \beta(1-\varepsilon)]^2$, then it decreases with the share parameters of physical and time inputs in the education sector, ε and $\beta(1-\varepsilon)$, respectively. The determination of λ_Δ differs from that of λ_K in several interesting ways. First, the share parameter of physical capital in the production sector α has no role in sizing λ_Δ, in contrast to its negative effect on λ_K. Second, a higher β, i.e. a weaker externality or a greater role of private investment in education, reduces λ_Δ but has no effect on λ_K. Third, increasing the share parameter of physical inputs in the education sector ϵ decreases λ_Δ but increases λ_K.

One important implication of the convergence analysis is to resolve the puzzling discrepancy between the empirically estimated and the model predicted speeds for standard assumptions about the value of α. In general, when the stationary convergence speed of output growth is equal to λ_Δ, it lies in the range $(0, \lambda_K)$. In particular, if the externality is very weak (i.e. Γ close to unity), then inequality and hence the economy will take infinitely many periods to reach the steady-state equilibrium, since $\lambda_\Delta \to 0$ as $\Gamma \to 1$.

The convergence analysis also has a strong implication for income comparison across economies. Should output growth converge quickly to a unique balanced rate across countries, income gaps would likely persist. However, if that convergence is slow and its pace varies across countries, some countries can maintain higher-than-average growth rates of per capita output for many periods (such as the growth miracle in East Asia for some decades), during which income ranking can switch. This strengthens the earlier discussion about the possibility of rapid or stagnant growth for income ranking change under particular initial conditions.

4. Simulation Results

Although the main characteristics of this dynamic system have been provided in the previous section, its quantitative implications are unclear. This section carries out quantitative assessments on the speed of convergence and provides numerical examples of income ranking change for plausible parameterization.

4.1. Speed of Convergence

To generate the annual growth rate and the annual speed of convergence, we assume that one period in this model corresponds to 20 years. The parameter configuration used in all the simulations below is given by $\alpha = 0.33$, $\varepsilon = 0.15$, $\eta = 0.9$, and $\rho = 0.8$, similar to those used in the literature. The values of α and ε imply that the annual speed of λ_K is 6.4%, roughly the same as was obtained from a neoclassical growth model. Given the value of ε, the parameter β becomes essential to determine $\lambda_\Delta = 1 - [\varepsilon + \beta(1-\varepsilon)]^2$, and hence $\lambda_\infty = \min\{\lambda_\Delta, \lambda_K\}$. Since little is known about the value of β, we allow it to vary in a wide range between 0.2 and 0.95. As β changes from case to case, so does the steady-state growth rate. To make things comparable, we generate the same annual rate of stationary growth around 2.1% in all cases by proper choice of D and μ_ξ in each case.[5] Moreover, initial inequality (Δ_0^2) and the initial ratio of physical to human capital (k_0) are set above their stationary values, and vary in magnitude across cases in tandem with the change in β.

The simulation results are reported in Figures 2-5 (corresponding to $\beta = 0.95$, 0.8, 0.5, or 0.2 and $D = 2$, 2.5, 2.5, or 0.95 respectively in Figures 2, 3, 4, and 5). In Figure 2 (with $\beta = 0.95$, $D = 2$, $\Delta_0^2 = 17$, and $\ln k_0 = 1$), the stationary solution is: $\Delta_\infty^2 = 12.02$, $\ln k_\infty = 0.146$, and the stationary annual speed of convergence implied by $\lambda_\infty = 1 - \Gamma^2$ equals 0.4%. Since initial inequality and the initial capital ratio are all set above their stationary values, the initial growth rate is below its stationary value, while the initial speed of convergence is above its stationary value. At such a

[5] By $\ln E(\xi) = \mu_\xi + \Delta_\xi^2/2$, the parameter μ_ξ affects the average learning ability, and hence plays a key role in generating plausible values of the growth rate even though it does not affect the analytical results in the previous section. In general, $\mu_\xi + \Delta_\xi^2/2 > 0$ is needed for the average learning ability $E(\xi)$ to be greater than 1 in order to have plausible human capital growth. We thus add terms that contain the factor $\mu_\xi + \Delta_\xi^2/2$ back to relevant equations used in the simulation.

low speed of convergence, it takes more than 30 periods for the growth rate of per capita output to reach 2.1% from below as seen in Figure 2a (Series 1). Human capital growth in Figure 2a (Series 2) first falls, owing to initial sharp declines in the capital ratio, and then rises to the stationary value, because of persistent declines in inequality.

In Figure 2b, Series 1 shows the falling annual speed of convergence to its stationary value, which is substantially lower than Series 2 that represents the constant annual speed of convergence 6.4% implied by λ_K. As noted earlier, the annual speed of convergence implied by λ_K corresponds to the one without income inequality. Thus, when inequality converges slowly with a large β, neglecting inequality would lead to a much higher speed of convergence as illustrated in Figure 2b.

Figures 3-5 share many common features with Figure 2. The main difference among the figures is the gap between the annual convergence speed implied by λ_t and the annual convergence speed implied by λ_K. When β becomes smaller, the speed of the convergence of output growth becomes higher (closer to the one without inequality from below). When $\beta = 0.8$ in Figure 3b, the annual convergence speed of output growth is around 2% with a stationary value at 1.9%, which accords well with the empirically estimated values in the literature. Even though β is reduced to 0.5 in Figure 4, the relevant speed of the convergence of output growth remains below the speed without inequality. The two convergence speeds become almost indistinguishable from each other when β takes a very small value (0.2) in Figure 5b.

Overall, the previously predicted speed of convergence (6-7% per year) in the literature is echoed here in cases without inequality, or with inequality plus an extremely small share parameter associated with the time investment in education. On the other hand, the empirically estimated speed of convergence at 2% per year in previous work is predicted here with highly plausible values of all share parameters in both sectors ($\alpha = 0.33$, $\beta = 0.8$, and $\varepsilon = 0.15$).[6]

[6] From this set of values, the share parameter of time inputs in the education sector $\beta(1-\varepsilon)$ equals 0.68, and the share parameter of average human capital (the strength of the externality) in the education sector $(1-\beta)(1-\varepsilon)$ equals 0.17.

4.2. Income Ranking Change

Now we set $\beta = 0.8$ that corresponds to the 1.9% annual rate of convergence, and use the same parameterization for other parameters. We report four cases in Table 1 to illustrate how income ranking can change within ten periods (200 years) when all the cases share the same balanced growth path.

In Case 1 of Table 1, a benchmark is constructed on the balanced path in all the ten periods, where the initial income level per worker is 1,000, the initial average human capital is 1648.253, and both initial inequality and the initial ratio of physical capital to effective labor are set at their respective steady-state levels. On this balanced path, the growth rate of per capita output equals 2.0876% and the measure of inequality Δ_{∞}^2 equals 3.2144. In the last period, per capita output reaches 41,225.

Case 2 starts with the same level of average human capital as in Case 1, but with lower inequality as well as a lower ratio of physical to human capital.[7] The lower initial physical to human capital ratio also leads to lower initial per capita output, 588. As argued earlier, the low initial inequality and the relative abundance of human capital are both conducive to growth, almost doubling the benchmark growth rate in the initial period. In addition, because of the slow convergence in the growth rate (initially much slower than the stationary level), the growth rate remains above its balanced rate throughout all the 10 periods. What happens to the income ranking? It only takes two periods (or 40 years) for the income level in Case 2 to exceed that in the benchmark from over 40% below in the initial period, creating miraculous overtaking. This example resembles the phenomenal growth of Germany and Japan during 1950-1990, after the war destroyed much of the physical capital and made many equally poor. The short length of time for this income overtaking is surprising in a two-sector growth model among economies that share the same balanced path, since, without considering inequality, it would only allow for *partial* catching up in income.

How high can the growth rate be for extremely low initial inequality and an extremely low initial

[7] The initial inequality in Case 2 is one-third of the benchmark level. According to international data on inequality, the highest Gini coefficients are above 0.6, more than three times as large as the lowest value (below 0.2).

ratio of physical to human capital? It could exceed 6.5%, miraculous indeed (3 times as high as the benchmark level). This extreme case may bring to light a possible contribution of initial conditions to the past 20-year extraordinary growth in China. Among many possible factors at the onset of this experience, two factors may be very important according to this model: extreme equality through land reform, free education and health care, and extreme lack of physical capital, particularly in terms of land per peasant. If this interpretation is correct, the particular combination of initial conditions could explain much of this 20-year rapid growth since the economic reform in China started in late 1970s, and may support similar growth performance in coming decades.

Case 3 is similar to Case 2, except that initial average human capital is reduced to 1236.190 and hence initial per capita output is lowered further to 441. The overtaking in income ranking, compared also with the benchmark, now takes 7 periods (or 140 years). What is interesting in this example is that Cases 2 and 3 have the same sequence of growth rates, although initial human capital and initial output all differ. This illustrates the earlier discussion on conditional convergence that proportionate changes in output and human capital alone have no effect on the growth rate of output.

Case 4 starts with lower average human capital, a lower ratio of physical to human capital, but higher inequality than does the benchmark. The combination of the initial conditions is chosen to mimic Latin American, where high inequality and development stagnation have long caught the attention in the literature. As expected, these initial conditions lead to stagnant growth: the growth rate in Case 4 remains below 1% for 3 periods (or 60 years) and slowly returns to normal. Starting with an initial output per capita much higher than in Cases 2 and 3, Case 4 falls from grace in all other periods. As a result, output per capita remains below the 1000 level for 3 periods (60 years), and takes almost 5 periods (100 years) to double.

5. Concluding Remarks

This paper has developed a two-sector growth model with life-cycle saving, human capital, and income inequality where global convergence is achieved through adjustments in both inequality and the ratio of physical to human capital. The results of this two-dimensional adjustment mechanism have gone well beyond a simple sum of those in the one-dimensional adjustment cases, and help resolve some puzzling issues. First, we have shown that in transition high initial inequality hinders output growth and physical capital growth but has an ambiguous effect on human capital growth, by taking into account the equilibrium feedback of inequality through the capital-effective labor ratio. Second, when inequality converges slowly, so does income growth; and the speed of convergence varies with initial conditions and over time. For plausible parameter values in both the production sector and the education sector, we can generate the 2% annual speed of convergence found in previous empirical studies. In particular, the variation of the rate of convergence depends on education regimes, and can thus explain why the OECD countries that have provided better public access to education have converged faster than other countries. Third, this enriched model allows rapid or stagnant growth so that income ranking can alter in short horizon among economies that share the same balanced path, complementing approaches that may generate income ranking change by considering country-specific uncertainties or multiple equilibria. Furthermore, the two-dimensional adjustment mechanism developed here appears to be a novel, yet realistic, feature, and may have potentials for wider applications.

Finally, the results in this paper may have some useful policy implications. Slow convergence in per capita output growth suggests that transitional dynamics should not be ignored opposed to steady-state features. Also, the message of the slow convergence to poor countries is not necessarily pessimistic, because the convergence speed is found as a function of initial states with the two-dimensional adjustment mechanism, other than a constant. For example, if inequality is reduced and human capital is increased by public policy, the growth rate of per capita income can rise immediately and stay high for many periods to come, due to slow convergence in the growth rate afterwards. During these transitional periods, the income gap with rich countries may be narrowed substantially, and miraculous overtaking in income is possible.

Appendix

The first-order conditions. The first-order conditions for (9) are:

$$z_t: \quad 1/z_t = \rho E_t[(\partial V_{t+1}/\partial h_{t+1})\beta(1-\varepsilon)h_{t+1}/(1-l_t-z_t)], \tag{30}$$

$$l_t: \quad w_t h_t/c_{1t} = 1/z_t, \tag{31}$$

$$q_t: \quad 1/c_{1t} = \rho E_t[(\partial V_{t+1}/\partial h_{t+1})\varepsilon h_{t+1}/q_t], \tag{32}$$

$$s_t: 1/c_{1t} = \eta(1+r_{t+1})/c_{2t+1}. \tag{33}$$

The envelope condition arises from differentiating (9) with respect to h_t

$$\partial V_t/\partial h_t = w_t l_t/c_{1t} + \rho E_t[(\partial V_{t+1}/\partial h_{t+1})\beta(1-\varepsilon)h_{t+1}/h_t]. \tag{34}$$

These conditions plus the budget constraints lead to the solution in (10)-(13). Q.E.D.

Proof of Proposition 1. Allowing k_t and Δ_t^2 to approach their limits in (15), (19), and (20) results in the unique steady-state growth rate of average physical capital, average human capital, and average output:

$$\ln(1+g_\infty) = \frac{\varepsilon\alpha}{1-\alpha(1-\varepsilon)}\Lambda + \Theta - \frac{\Gamma(1-\Gamma)(1-\alpha)}{[1-\alpha(1-\varepsilon)]}\frac{\Delta_\infty^2}{2}. \tag{35}$$

The global stability of this system is obvious, since under $0 < \alpha < 1$, $0 < \varepsilon < 1$, and $0 < \Gamma < 1$, both $\alpha^t(1-\varepsilon)^t$ and Γ^{2t} approach zero as t rises. Differentiating g_∞ with respect to Δ_∞^2, γ_q, and γ_s, respectively, gives their effects on g_∞. Also, substituting k_∞ into (6) and (7) produces w_∞ and r_∞. The relations of k_∞, w_∞, and r_∞ with Δ_∞^2 are derived from (6), (7), and (24). Q.E.D.

Proof of Proposition 2. For part (i), high k_0 implies high k_t for $t > 0$ by (23). The rest of part (i) follows since $\partial w_t/\partial k_t > 0$, $\partial r_t/\partial k_t < 0$, $\partial g_{K,t}/\partial k_t < 0$, $\partial g_{H,t}/\partial k_t > 0$, and $\partial g_{Y,t}/\partial k_t < 0$ by (6), (7), (15), (19), and (20).

For part (ii), $\partial \ln k_t/\partial \Delta_0^2 > 0$ by (23). Also, from (15) and (20), $\partial g_{K,t}/\partial \ln k_t < 0$ and $\partial g_{Y,t}/\partial \ln k_t < 0$ for $t > 0$. Therefore, high Δ_0^2 leads to low $g_{K,t}$ and $g_{Y,t}$ indirectly via $\ln k_t$

for $t > 0$. Furthermore, from (15), (20), and (22), Δ_0^2 has a direct negative (zero) effect on $g_{Y,t}$ ($g_{K,t}$) via Δ_t^2 for $t \geq 0$. These establish the results in regard to the negative effects of Δ_0^2 on $g_{K,t}$ and $g_{Y,t}$. By (19), $\partial g_{H,0}/\partial \Delta_0^2 < 0$. To see the effect of Δ_0^2 on $g_{H,t}$ for $0 < t < \infty$, we rewrite (19) by using (22) and (23) as

$$\ln(1+g_{H,t}) = \Theta + \frac{\varepsilon\alpha[1-\alpha^t(1-\varepsilon)^t]}{1-\alpha(1-\varepsilon)}\Lambda - \frac{[1-\alpha+\varepsilon\alpha^{t+1}(1-\varepsilon)^t]\Gamma(1-\Gamma)}{1-\alpha(1-\varepsilon)}\left(\frac{\Delta_\infty^2}{2}\right)$$

$$+\frac{\Gamma(1-\Gamma)}{2}\left(\frac{\varepsilon\alpha[\Gamma^{2t}-\alpha^t(1-\varepsilon)^t]}{\Gamma^2-\alpha(1-\varepsilon)} - \Gamma^{2t}\right)(\Delta_0^2 - \Delta_\infty^2)$$

$$+\varepsilon\alpha^{t+1}(1-\varepsilon)^t \ln k_0. \tag{36}$$

By (36), $\partial g_{H,t}/\partial \Delta_0^2$ is signed by $\varepsilon\alpha[\Gamma^{2t}-\alpha^t(1-\varepsilon)^t]/[\Gamma^2-\alpha(1-\varepsilon)] - \Gamma^{2t}$, which can be positive, negative, or zero for $0 < t < \infty$. By (6), (7), and (23), the effects of Δ_0^2 on w_t and r_t follow, since $\partial k_t/\partial \Delta_0^2 > 0$ as shown above, $\partial w_t/\partial k_t > 0$ by (6), and $\partial r_t/\partial k_t < 0$ by (7). Q.E.D.

Proof of Proposition 3. From the production function, we have $\ln Y_t = \ln D + \alpha \ln k_t + \ln H_t$ where $\ln H_t = \mu_t + \Delta_t^2/2$. Thus, in the initial period, the income gap equals $\ln Y_0 - \ln Y_0' = \alpha(\ln k_0 - \ln k_0') < 0$ under $k_0 < k_0'$ and $H_0 = H_0'$. So $Y_0 < Y_0'$. For later periods, we formulate the income gap as:

$$\ln Y_t - \ln Y_t' = \alpha(\ln k_t - \ln k_t') + \mu_t - \mu_t' + (\Delta_t^2 - \Delta_t'^2)/2$$

Using (22), (23) and (25), one can express all the three gaps in the right-hand side of the above equation as functions of initial conditions:

$$\ln Y_t - \ln Y_t' = \Pi_1(\Delta_0^2 - \Delta_0'^2) + \Pi_2(\ln k_0 - \ln k_0') + \ln H_0 - \ln H_0',$$

where

$$\Pi_1 = \frac{-\Gamma(1-\alpha)}{(1+\Gamma)[1-\alpha(1-\epsilon)]} + \frac{\Gamma(1-\Gamma)\Gamma^{2t}}{1-\Gamma^2} + \left[\frac{\alpha\Gamma(1-\Gamma)}{\Gamma^2-\alpha(1-\epsilon)}\right]\times$$

$$\left[\Gamma^{2t} - \alpha^t(1-\epsilon)^t - \frac{\epsilon\Gamma^{2t}}{1-\Gamma^2} + \frac{\epsilon\alpha^t(1-\epsilon)^t}{1-\alpha(1-\epsilon)}\right],$$

$$\Pi_2 = \alpha^{t+1}(1-\epsilon)^t + \frac{\epsilon\alpha[1-\alpha^t(1-\epsilon)^t]}{1-\alpha(1-\epsilon)} > 0.$$

If inequality were missing (i.e. $\Delta_t = \Delta'_t = 0$ for all t) then it would be impossible to have $\ln Y_t > \ln Y'_t$ in any t, under $H_0 = H'_0$ and $k_0 < k'_0$. The same result would also hold if physical capital were missing (i.e. $\alpha = 0$), since under $\alpha = 0$, $\Pi_2 = 0$ but $\Pi_1 = -\Gamma(1-\Gamma^{2t})/(1+\Gamma) < 0$. With inequality and two types of capital, however, for a large enough t, $\Pi_1 < 0$ holds, so that the economy starting with lower initial inequality as well as a lower initial ratio of physical to human capital may have higher output per capita as long as $\Pi_1(\Delta_0^2 - \Delta'^2_0) > \Pi_2(\ln k'_0 - \ln k_0)$. Q.E.D.

Proof of Proposition 4. From (29) the speed of convergence λ_t depends on both (k_0, Δ_0^2) and $(k_\infty, \Delta_\infty^2)$, and varies with t. For the values of the stationary speed, there are four cases differentiated by whether $\Gamma^2 > \alpha(1-\varepsilon)$ or $\Gamma^2 < \alpha(1-\varepsilon)$, and by whether sign $d_{t+1} =$ sign d_t or sign $d_{t+1} = -$ sign d_t. Suppose first that sign $d_{t+1} =$ sign d_t for all t. If $\Gamma^2 > \alpha(1-\varepsilon)$ then $\alpha^t(1-\varepsilon)^t/\Gamma^{2t}$ approaches zero as t rises. Dividing both the numerator and the denominator of the right-hand side of (29) by Γ^{2t}, we thus obtain $\lim_{t\to\infty}\lambda_t = \lambda_\Delta = 1-\Gamma^2$ under $\Gamma^2 > \alpha(1-\varepsilon)$. Note also that $\Gamma^2 > \alpha(1-\varepsilon)$ implies $\lambda_\Delta < \lambda_K$. Analogously, we can establish $\lim_{t\to\infty}\lambda_t = \lambda_K = 1-\alpha(1-\varepsilon)$ and $\lambda_K < \lambda_\Delta$ under $\Gamma^2 < \alpha(1-\varepsilon)$. Thus, $\lambda_\infty = \min\{\lambda_\Delta, \lambda_K\}$ if sign $d_{t+1} =$ sign d_t.

Suppose now that sign $d_{t+1} = -$ sign d_t for all t. Paralleling the argument above, $\lim_{t\to\infty}\lambda_t = \lambda'_\Delta = 1+\Gamma^2$ under $\Gamma^2 > \alpha(1-\varepsilon)$, and $\lim_{t\to\infty}\lambda_t = \lambda'_K = 1+\alpha(1-\varepsilon)$ under $\Gamma^2 < \alpha(1-\varepsilon)$. Since both λ'_Δ and λ'_K are greater than one, the system would diverge away from its steady-state growth path if sign $d_{t+1} = -$ sign d_t for all t. This contradicts Proposition 1. Therefore, sign $d_{t+1} = -$ sign d_t can only occur for a finite number of periods and in the long run the stationary speed of convergence is either λ_Δ or λ_K, whichever is lower, under sign $d_{t+1} =$ sign d_t. Q.E.D.

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Table 1. Simulation Results: Output (per capita) and Annual Growth (%)*

Parameters: $\alpha = 0.33$, $\beta = 0.8$, $\epsilon = 0.15$, $\eta = 0.9$, $\rho = 0.8$, $D = 2.5$								
Period (20 years each)	Case 1. Benchmark		Case 2. Low (k_0, Δ_0^2)		Case 3. Low (k_0, Δ_0^2, H_0)		Case 4. Low (k_0, H_0) & high Δ_0^2	
	$\Delta_0^2 = 3.2144$		$\Delta_0^2 = 1.07147$		$\Delta_0^2 = 1.07147$		$\Delta_0^2 = 9.6432$	
	$k_0 = 0.24222$		$k_0 = 0.04844$		$k_0 = 0.04844$		$k_0 = 0.18166$	
	$H_0 = 1648.253$		$H_0 = 1648.253$		$H_0 = 1236.190$		$H_0 = 1236.190$	
	output	growth	output	growth	output	growth	output	growth
0	1000	2.0876	588	4.1695	441	4.1695	682	0.8204
1	1512	2.0876	1331	3.0262	998	3.0262	803	0.6713
2	2285	2.0876	2416	2.5963	1812	2.5963	918	0.9577
3	3455	2.0876	4034	2.3995	3026	2.3995	1111	1.2651
4	5222	2.0876	6482	2.2916	4861	2.2916	1429	1.5083
5	7894	2.0876	10198	2.2251	7648	2.2251	1927	1.6848
6	11934	2.0876	15836	2.1815	11877	2.1815	2692	1.8090
7	18040	2.0876	24384	2.1520	18287	2.1520	3853	1.8953
8	27271	2.0876	37328	2.1319	27996	2.1319	5609	1.9550
9	41225	2.0876	56920	2.1181	42690	2.1181	8261	1.9963

* In all cases, the steady-state inequality (Δ_∞^2) equals 3.2144 and the steady-state ratio of physical capital to effective labor (k_∞) equals 0.24222.

Figure 1a. Initial Gini Coefficients and Per Capita Income Growth 1970-1985 (44 Countries)

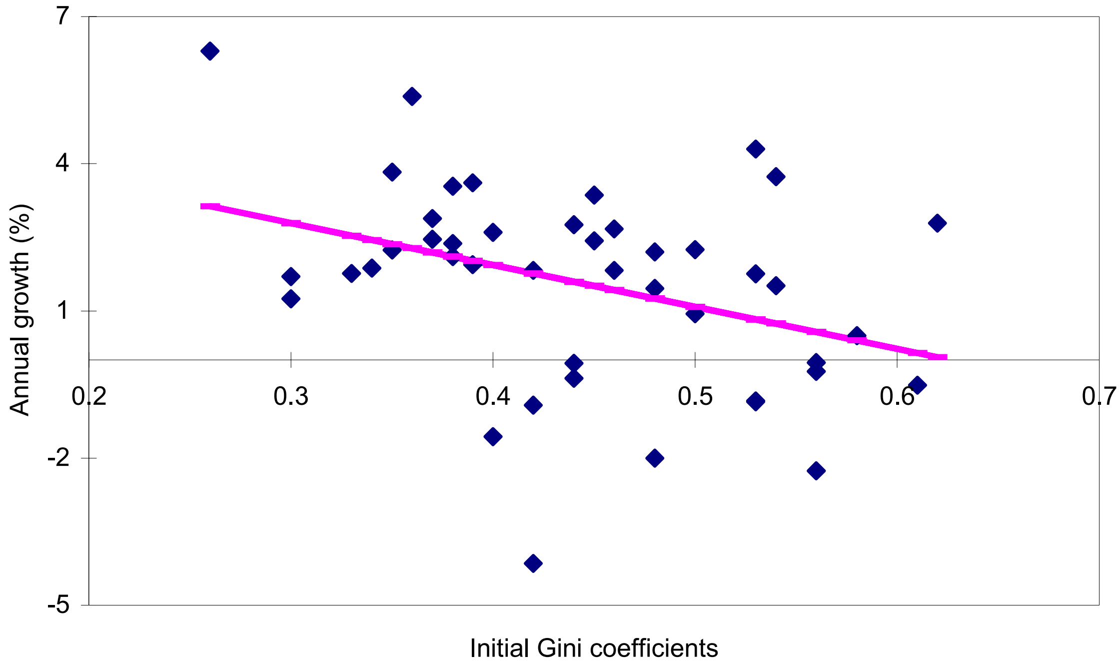


Figure 1b. Initial Gini Coefficients and Average Human Capital Growth 1970-1985 (44 Countries)

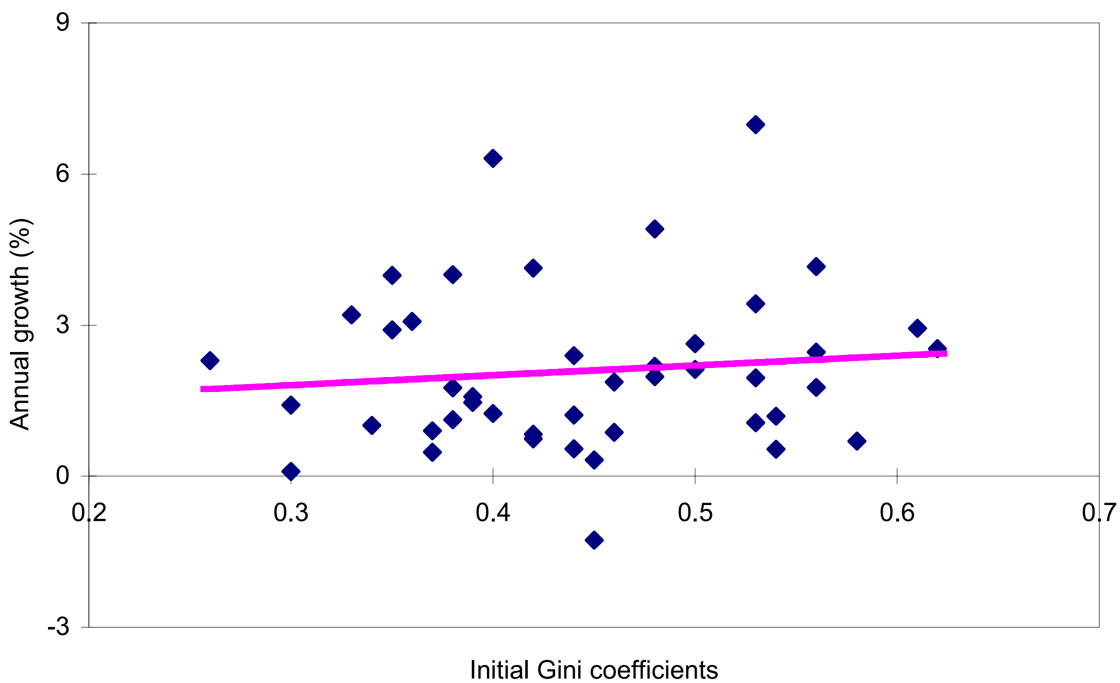


Figure 2a. Growth of Per Capita Income (Series 1) and Human Capital (Series 2)

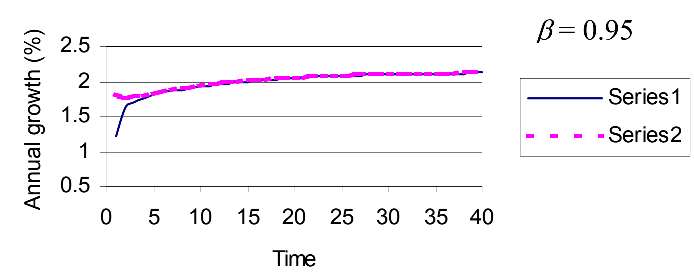


Figure 2b. Speed of Convergence (Series 1: for Income Growth; Series 2: without Inequality)

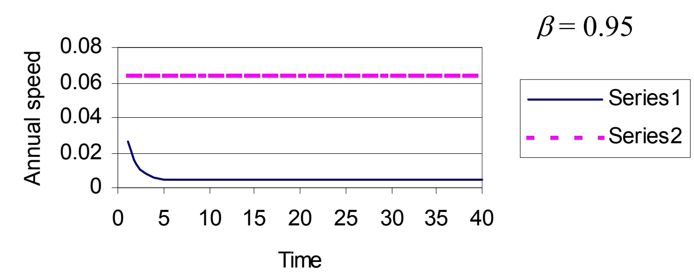


Figure 3a. Growth of Per Capita Income (Series 1) and Human Capital (Series 2)

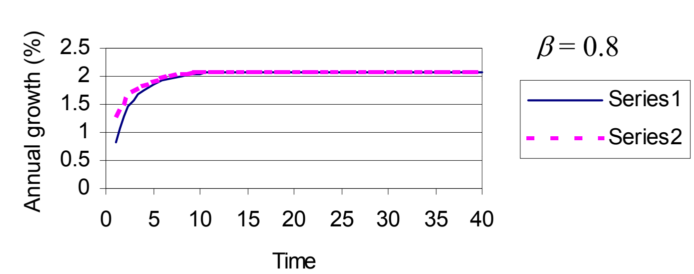
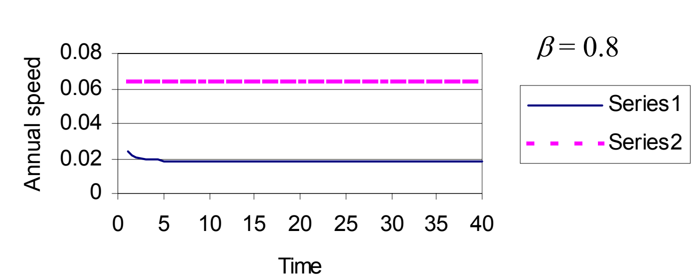


Figure 3b. Speed of Convergence (Series 1: for Income Growth; Series 2: without Inequality)



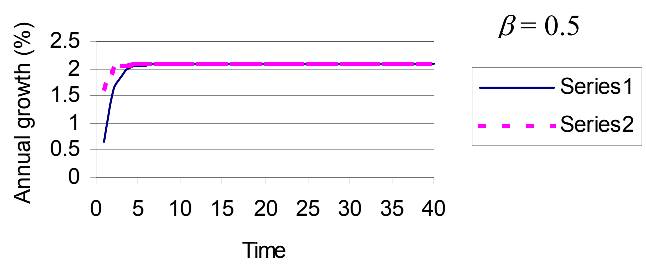


Figure 4a. Growth of Per Capita Income (Series 1) and Human Capital (Series 2)

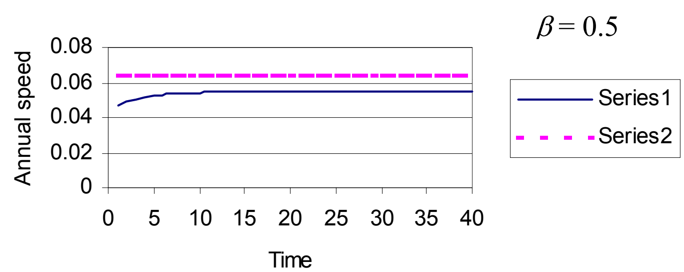


Figure 4b. Speed of Convergence (Series 1: for Income Growth; Series 2: without Inequality)

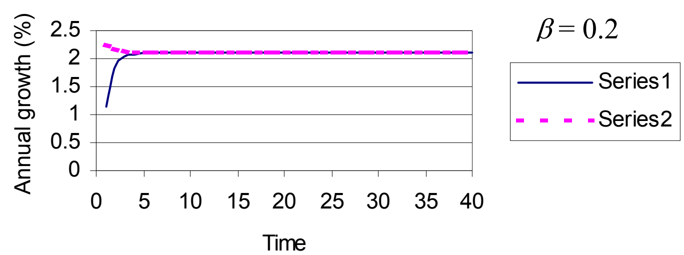


Figure 5a. Growth of Per Capita Income (Series 1) and Human Capital (Series 2)

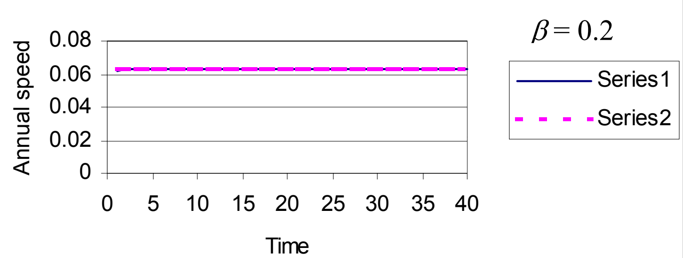


Figure 5b. Speed of Convergence (Series 1: for Income Growth; Series 2: without Inequality)