The eigensystem vector autogression model (AMW DISCUSSION DRAFT)[*]

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Abstract

This article introduces the eigensystem vector autogression (EVAR) model. The EVAR transforms a standard VAR into a system of seemingly-unrelated univariate first-order complex autoregressions. These individual components offer an alternative perspective for interpreting and adjusting the VAR. Two illustrations of the latter are provided, i.e.: (1) improving on VAR forecasts via restricting negligible EVAR coefficients to zero; and (2) the potential for correcting the biases of estimated VAR coefficients via adjusting the dominant components of the EVAR.

JEL: C002, C32, C52

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1 Introduction

This article introduces the eigensystem vector autoregression (EVAR) model. The EVAR uses the eigensystem decomposition of the companion form for a standard reduced-form VAR to transform that VAR and its data into a system of seemingly-unrelated univariate first-order complex autoregressions. Equivalently, the EVAR is itself a VAR with a diagonal matrix of complex coefficients, complex data, and complex innovations with a non-diagonal covariance matrix.

The motivation for such a transformation is to provide a component perspective of the original VAR that may be more amenable to economic interpretation and for making adjustments to the VAR depending on the particular application. Specifically regarding interpretation, the coefficients and innovations from the individual autoregressions in the EVAR may readily be assessed for their contribution to the dynamic multipliers and innovation variances of the VAR. Regarding model adjustment, two illustrations of the latter are provided, i.e.: (1) improving on VAR forecasts via restricting negligible EVAR coefficients to zero; and (2) the potential for correcting the biases of estimated VAR coefficients via adjusting the dominant components of the EVAR

The motivation for the article itself is that, to the best of the author's knowledge, the equivalent of the EVAR transformation and its potential applications does not appear to have been fully explored in the literature. To clarify, the calculation of companion-form eigenvalues is certainly a standard step for assessing the stability of VAR models and the dynamic multipliers of their innovations, as outlined in econometrics textbooks. However, the employment of the full eigensystem in conjunction with its transformed data does not appear common.

The only directly-related example found in a reasonably thorough search of the economic/econometric literature was Schoonbeek (1987). That article investigates whether the estimated coefficients of an economic model written in VAR form may be described using a small subset of eigenvectors, and what the sensitivity of those eigenvectors are to the coefficients of the original VAR model. While that is consistent with elements of sections 3 and 4 of the present article, Schoonbeek (1987) does not consider the transformed data. Only one directly-related example was found in the physical sciences literature; i.e. the pair of companion articles by Neumaier and Schneider (2001a and b). Those articles set out a framework similar to the EVAR for decomposing the VAR and its data. However, the applications focus initially on a computationally-efficient method for determining the appropriate lag length of VAR models (which is already well-established in the econometrics literature), and then the interpretation of the VAR components (e.g. the relative dynamic importance of the transformed components and the estimation of approximate confidence intervals for the eigenvectors). The potential for using the EVAR and its data to adjust the VAR for subsequent use is not explicitly considered (although a forecasting role is implied, given the eigenvectors are discussed as a multivariate analogue to principal oscillation patterns from univariate data within the field of geophysics).

Note that, although there are certain analogies, the EVAR is distinct from principal components analysis (the eigensystem decomposition of the covariance matrix, e.g. Joliffe 2002), forecasting using principal components (effectively using the latter to approximate a finite-order moving-average model e.g. Stock

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and Watson 2002), and the factor-augmented VAR (using time series data derived from principle components analysis in conjunction with other data within a VAR, e.g. Bernanke, Boivin and Eliasz 2005). The EVAR has a closer analogy with the generalized dynamic-factor model Forni, Hallin, Lippi and Reichlin 2000. The theoretical framework in that article allows for unobserved common factors with an infinite-order moving-average structure and so, in principle, accommodates VAR processes. However, the model cannot be estimated on the basis of traditional methods. Conversely, the EVAR simply uses ordinary least squares for the VAR and the analogue complex least squares for the EVAR.

The article is structured as follows. Section 2 briefly outlines the notation of the standard reduced-form VAR and its companion form. Section 3 shows how the companion form of the VAR may be transformed into the EVAR. Section 4 discusses the interpretation of the EVAR components with respect to the VAR. Section 5 discusses the potential for using the EVAR to effectively reduce overparametrisation in the original VAR, and the concept is illustrated with a simple forecasting application. Section 6 comments briefly on the potential for using the EVAR to correct for biases in the estimated coefficients of VAR models. Section 7 concludes.

2 The standard VAR model

As noted in standard textbooks (e.g. Hamilton (1994) pp. 257–258),[1] the VAR model for N variables is:

$$x_t = \alpha + \sum_{p=1}^{P} \Phi_p x_{t-p} + \varepsilon_t \tag{1}$$

where x_t is an $N \times 1$ vector of data at time t, α is an $N \times 1$ constant vector, x_{t-p} is an $N \times 1$ vector of data at time $t - p$, Φ_p is an $N \times N$ matrix of coefficients relating x_t to x_{t-p}, P is the order of the VAR (i.e. the number of lags of x_t included on the right-hand side of the equation), and ε_t is an $N \times 1$ vector of residuals with $\mathbb{E}\left[\varepsilon_t\right] = 0$ and $\mathbb{E}\left[\varepsilon_t \varepsilon_s'\right] = \Omega$ if $t = s$ and zero otherwise. The length of the original time series is T, but the time index t ranges from $P + 1$ to T after allowing for lags. The VAR may be expressed equivalently relative to its mean with the transformation $y_t = x_t - \mu$, where $\mu = \left(I_N - \sum_{p=1}^{P} \Phi_p\right)^{-1} \alpha$,

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[1] Sims (1980) is appropriately cited as the seminal article for the literature on VARs.

with I_N the $N \times N$ identity matrix. Hence, equation 1 becomes:

$$y_t = \sum_{p=1}^{P} \Phi_p y_{t-p} + \varepsilon_t \tag{2}$$

where the variables and parameters are as previously defined.

Again following standard practice (e.g. Hamilton (1994) p. 259), the VAR model on equation 2 may be rewritten in its companion form as an NP-variable first-order VAR, i.e.:

$$Y_t = FY_{t-1} + E_t \tag{3}$$

where $Y_t = \left[y_t', y_{t-1}', \ldots, y_{t-P+1}' \right]'$ (an $NP \times 1$ vector); $Y_{t-1} = \left[y_{t-1}', y_{t-2}', \ldots, y_{t-P}' \right]'$ (an $NP \times 1$ vector);

$$F = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \cdots & \Phi_{P-1} & \Phi_P \\ I_N & 0_N & 0_N & \cdots & 0_N & 0_N \\ 0_N & I_N & 0_N & \cdots & 0_N & 0_N \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_N & 0_N & 0_N & \cdots & I_N & 0_N \end{bmatrix} \tag{4}$$

with 0_N an $N \times N$ matrix of zeros (so F is an $NP \times NP$ matrix); $E_t = [\varepsilon_t', 0', \ldots, 0']'$ with $0'$ an $N \times 1$ vector of zeros (so E_t is an $NP \times 1$ vector), and $\mathbb{E}\left[E_t E_s'\right] = 0$ if $t \neq s$ and Θ if $t = s$ with:

$$\Theta = \begin{bmatrix} \Omega & 0_N & \cdots & 0_N \\ 0_N & 0_N & \cdots & 0_N \\ \vdots & \vdots & \ddots & \vdots \\ 0_N & 0_N & \cdots & 0_N \end{bmatrix} \tag{5}$$

The eigenvalues of F are traditionally used to assess whether the VAR is stable; i.e. if each eigenvalue lies within the unit circle then the effect of ε_t will decay to zero. If so, the VAR may then be used without any issues for typical economics applications, such as forecasting and generating impulse responses, etc.

Another avenue is to undertake the full eigensystem decomposition of F and assess if that might offer a useful perspective for the VAR. That proceeds in the following section.

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3 The EVAR model

The EVAR is obtained from the eigensystem decomposition of the companion form of the VAR. That is, rewrite F in eigensystem form; i.e.:

$$F = VDV^{-1} \tag{6}$$

where D is an $NP \times NP$ diagonal matrix of eigenvalues and V is an $NP \times NP$ matrix containing the associated eigenvectors in columns (i.e. so V is an $1 \times NP$ row vector of $NP \times 1$ column vectors V_j). Substitute VDV^{-1} for F in equation 3, and rearrange, i.e.:

$$
\begin{align}
Y_t &= VDV^{-1}Y_{t-1} + E_t \tag{7a} \\
V^{-1}Y_t &= DV^{-1}Y_{t-1} + V^{-1}E_t \tag{7b} \\
Z_t &= DZ_{t-1} + \eta_t \tag{7c}
\end{align}
$$

where $Z_t = V^{-1}Y_t$ and $\eta_t = V^{-1}E_t$ (so both both Z_t and η_t are $NP \times 1$ vectors). Hereafter, equation 7 is denoted the eigensystem VAR or the EVAR for short.

The property that makes the EVAR potentially useful is that, given D is a diagonal matrix by definition, each line of equation 7c represents an individual component of the original VAR. Each component is also automatically in the form of univariate first-order autoregression or AR(1) model, potentially involving complex numbers if F contains complex eigenvalues. Complex values within the EVAR will always appear in tandem with their complex conjugate values. Those pairs may be processed in tandem within the EVAR, which results in all real values when transforming back to the original VAR space. Alternatively, processing complex values individually within the VAR will result in complex values when transformed back to the original VAR space. The apppropriate real components are then used in subsequent analysis, as will be seen in the forecasting application of section 5.2.

In summary, each individual complex AR(1) or CAR(1) model is:

$$Z_{j,t} = D_j Z_{j,t-1} + \eta_{j,t} \tag{8}$$

where j ranges from 1 to NP for the EVAR. CAR models are already used in the physical sciences for applications such as pattern recognition, e.g. Umeyama (1998) and Nakatani, Sasaki, Iaguni and Maeda (1999), and weather forecasting, e.g. Gu and Jiang (2005). The latter article also establishes that the least squares estimate of a CAR(1) is simply the complex analogue of the least squares estimate for the AR(1) model. Hence, the autoregressive parameter D_j for the

j^{th} CAR(1) is:

$$D_j = Z_{j,0} Z_{j,-1}^\dagger \left(Z_{j,-1} Z_{j,-1}^\dagger \right)^{-1} \tag{9}$$

where † denotes the complex conjugate transpose operator (i.e. complex conjugation of each matrix entry and then a matrix transposition), $Z_{j,0}$ is the $1 \times T$ vector of values $Z_{j,t}$, and $Z_{j,-1}$ is the $1 \times T$ vector of lagged valued $Z_{j,t-1}$.[2] It can also be confirmed that the complex least squares estimate for any other pair of time series Z_j and Z_k with $j \neq k$ is zero; i.e. $Z_{j,-1} Z_{k,0}^\dagger \left(Z_{j,-1} Z_{k,-1}^\dagger \right)^{-1} = 0$.

For later reference, the version of the EVAR with its full data matrix is:

$$Z_0 = D Z_{-1} + \eta_0 \tag{10}$$

where $Z_0 = \left[Z'_{P+1}, \ldots, Z'_T \right]'$, $Z_{-1}^R = \left[Z'_P, \ldots, Z'_{T-1} \right]'$, and $\eta_0 = \left[\eta'_{P+1}, \ldots, \eta'_T \right]'$, all $NP \times (T - P)$ matrices. The full matrix analogue of the individual CAR(1) estimates is then:

$$D = Z_0 Z_{-1}^\dagger \left(Z_{-1} Z_{-1}^\dagger \right)^{-1} \tag{11}$$

The residuals η_0 from the EVAR have the covariance matrix:

$$\Psi = \mathbb{E}\left[\eta_t \eta_t^\dagger \right] = \mathbb{E}\left[V^{-1} E_t E_t^\dagger \left(V^{-1} \right)^\dagger \right] = V^{-1} \mathbb{E}\left[E_t E_t' \right] \left(V^{-1} \right)^\dagger \tag{12a}$$

which is not diagonal given that both $\mathbb{E}\left[E_t E_t' \right]$ and V are non-diagonal matrices. Hence, while the EVAR is a system of individual CAR(1) models, those models are not independent. The non-zero covariance of the residuals means the EVAR is actually a system of seemingly-unrelated CAR(1) models. Equivalently, the EVAR may be viewed and applied as in equations 7c and 10; i.e. as a VAR with a diagonal matrix of complex coefficients, complex data, and complex innovations with a non-diagonal covariance matrix.

4   Interpreting the EVAR model

Having transformed the N-variable VAR with P lags into a system of individual NP CAR(1) models, the latter may be readily interpreted for their contributions to the dynamics of the VAR. Essentially, the expected dynamics of each

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[2]If the series $Z_{j,t}$ is real, then equation 9 becomes the standard OLS estimate for AR(1) models; i.e $D_j = Z_{j,t-1} Z'_{j,t} \left(Z_{j,t-1} Z'_{j,t-1} \right)^{-1}$. Note that the latter is the transpose of the typical OLS expression because the VAR specification implicitly defines the time dimension of the data along rows (while AR specifications typically define the time dimension down columns).

component of the EVAR follows the standard interpretation of univariate AR models from econometrics textbooks (e.g. Hamilton (1994) pp. 13–18), and those dynamics are reflected in Y_t (and so y_t) via the EVAR's eigenvectors.

Positive real eigenvalues D_j (assuming they are less than 1) imply that non-zero values of $Z_{j,t}$ will be expected to undergo an exponential decay over time, i.e. $\mathbb{E}_t\left[Z_{j,t+\tau}\right] = D_j^\tau Z_{j,t}$, with a half-life $\frac{-\log(2)}{\log(D_j)}$. Because $Y_t = VZ_t$, the effect of the expected dynamics of $Z_{j,t}$ will be reflected in the expected variables Y_t according to the linear combination given by the eigenvector V_j; i.e. $\mathbb{E}_t\left[Y_{t+\tau}(j)\right] = \mathbb{E}_t\left[Z_{j,t+\tau}\right]V_j$ where $Y_t(j)$ is the component of Y_t due to $Z_{j,t}$. Negative real eigenvalues D_j (assuming $|D_j| < 1$) have the same interpretation except the sign of $Z_{j,t}$ alternatives with each increment of time.

A pair of complex conjugate eigenvalues D_j and D_{j+1} (assuming they are within the unit circle) implies that non-zero values of $Z_{j,t}$ and its complex conjugate $Z_{j+1,t}$ will follow a sinusoidal cycle with an exponential decay in its magnitude. Explicitly, the eigenvalues may be re-expressed as $D_j = r\left[\cos(\theta) + i\sin(\theta)\right]$ and $D_{j+1} = r\left[\cos(\theta) - i\sin(\theta)\right]$ with $r = \sqrt{\left[\mathrm{Re}(D_j)\right]^2 + \left[\mathrm{Im}(D_j)\right]^2}$ and $\theta = \arccos\left(\frac{\mathrm{Re}(D_j)}{r}\right)$. Then:

$$\begin{align}
\mathbb{E}_t\left[Z_{j,t+\tau} + Z_{j+1,t+\tau}\right] &= D_j^\tau Z_{j,t} + D_{j+1}^\tau Z_{j+1,t} \tag{13a} \\
&= 2r^\tau\left[\mathrm{Re}(Z_{j,t})\cos(\theta\tau) - \mathrm{Im}(Z_{j,t})\sin(\theta\tau)\right] \tag{13b}
\end{align}$$

with a half-life for r of $\frac{-\log(2)}{\log(r)}$ and a period of $\frac{2\pi}{\theta}$. Similarly:

$$\begin{align}
-i\mathbb{E}_t\left[Z_{j,t+\tau} - Z_{j+1,t+\tau}\right] &= -i\left(D_j^\tau Z_{j,t} - D_{j+1}^\tau Z_{j+1,t}\right) \tag{14a} \\
&= 2r^\tau\left[\mathrm{Im}(Z_{j,t})\cos(\theta\tau) + \mathrm{Re}(Z_{j,t})\sin(\theta\tau)\right] \tag{14b}
\end{align}$$

The expected component of Y_t due to the jth and $(j+1)^{\text{th}}$ EVAR components, i.e $\mathbb{E}_t\left[Y_{t+\tau}(j) + Y_{t+\tau}(j+1)\right]$, is:[3]

$$\begin{align}
&\mathbb{E}_t\left[Z_{j,t+\tau}V_j + Z_{j+1,t+\tau}V_{j+1}\right] \tag{15a} \\
=\ & D_j^\tau Z_{j,t}V_j + D_{j+1}^\tau Z_{j+1,t}V_{j+1} \tag{15b} \\
=\ & 2r^\tau\left[\mathrm{Re}(Z_{j,t})\cos\theta\tau - \mathrm{Im}(Z_{j,t})\sin\theta\tau\right]\mathrm{Re}(V_{j,t}) \\
& +2r^\tau\left[\mathrm{Im}(Z_{j,t})\cos\theta\tau + \mathrm{Re}(Z_{j,t})\sin\theta\tau\right]\mathrm{Im}(V_{j,t}) \tag{15c} \\
=\ & \left(D_j^\tau Z_{j,t} + D_{j+1}^\tau Z_{j+1,t}\right)\mathrm{Re}(V_{j,t}) \\
& -i\left(D_j^\tau Z_{j,t} - D_{j+1}^\tau Z_{j+1,t}\right)\mathrm{Im}(V_{j,t}) \tag{15d}
\end{align}$$

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[3]Rainville and Bedient (1981) pp. 256-261 gives an analogous result for the continuous time case. The case of repeated eigenvalues is not considered here because they will almost certainly not arise in empirical work. However, it could readily be allowed for if required.

The EVAR residuals are also a critical component of the systems overall dynamics, because they determine the extent to which noise can affect the expected path of the EVAR and VAR variables. For the real eigenvalues, shocks to $Z_{j,t}$ will be imparted to the original variables as $E_t(j) = \eta_{j,t} V_j$. For the pairs of complex conjugate eigenvalues, shocks to $Z_{j,t}$ and $Z_{j+1,t}$ will be imparted to the original variables as $E_t(j, j+1) = \eta_{j,t} V_j + \eta_{j+1,t} V_{j+1}$.

5 Application: Forecasting with subset EVARs

5.1 Background

The following assumes that the VAR order has already been selected using an appropriate method. Choosing the order for a VAR model is an important step,[4] and has accordingly received much attention in the literature (e.g. Lütkepohl (2007) chapter 4 discusses likelihood ratio and information criteria (IC) tests).

With the VAR order selected, the focus is then on how the VAR may potentially be refined further for the particular application. In the terminology of Lütkepohl (2007) section 5.2.8, that amounts to creating a subset VAR. Brüggemann (2004) pp. 10-12 details more of the literature on subset VARs, and pp. 12-24 discusses the methods that Brüggemann (2004) investigates.[5] The common feature is that zero restrictions are made to one or more of the originally-estimated coefficients. However, other authors argue against subset VARs on the grounds that the final model risks misspecification.[6] Another standard approach is therefore to estimate and retain all parameters up to a the pre-estimated lag length, even though that comes at the expense of over-parametrisation.[7]

The EVAR may offer a means for effectively reducing overparametrisation in VAR models in a way that is less likely to result in model misspecification.

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[4] From Enders (2004), p. 281: "Appropriate lag length selection can be critical. If p [the lag length] is too small, the model is misspecified; if p is too large, degrees of freedom are wasted."

[5] For example, the system strategies include System Full Search (finding the minimum IC for all $2^{N^2 P}$ subsets), System Sequential Elimination of Regressors (step removal of regressors that give the largest fall in IC under no further reductions are possible), and System Test Procedure (restricting insignificant coefficients to zero based on their associated t-statistics). Analogues of the system strategies may also be applied equation-by-equation within the system.

[6] Again from Enders (2004), p. 270: "Improperly imposing zero restrictions may waste important information. Moreover, the regressors are likely to be highly colinear so that the t-tests on individual coefficients are not reliable guides for paring down the model."

[7] Again from Enders (2004), p. 270: "Unquestionably, a VAR will be overparametrized in that many of these coefficient estimates will be insignificant."

That is, the eigensystem from the decomposition in section 3 indicates which components are most important to the dynamic multipliers of the system, while the covariance matrix of residuals indicates which components contribute the largest innovations. Considered in combination, the "dynamic importance" of each EVAR component (i.e. its eigensystem and covariance dynamics) may be evaluated when specifying an alternative model. Once the appropriate subset EVAR is determined, it may used directly or transformed back to the VAR space for its intended use.

As an illustration of the concept, the following section applies the EVAR to a simulating forecasting exercise with a relatively naive rule for eliminating the "non-important" components of the VAR; i.e. testing the statistical significance of the smallest coefficient.

5.2 A forecasting illustration via simulation

To simulate VAR data, equation 1 with $N = 2$ and $P = 2$ is used with the following parameters:

$$\alpha = \left[\begin{array}{c} 0.02 \\ 0.03 \end{array} \right]; \quad \Phi_1 = \left[\begin{array}{cc} 0.5 & 0.1 \\ 0.4 & 0.5 \end{array} \right]; \quad \Phi_2 = \left[\begin{array}{cc} 0 & 0 \\ 0.25 & 0 \end{array} \right]; \quad \Omega = \left[\begin{array}{cc} 0.09 & 0 \\ 0 & 0.09 \end{array} \right]$$

$$(16)$$

This model is from Lütkepohl (2007) p.137, with the one difference being $\Omega_{2,1} = 0.09$ instead of 0.04.[8] Note that the Cholesky decomposition of Ω is chol$(\Omega) = $ diag$[0.3, 0.3]$ and the innovations to the model are chol$(\Omega) [N(0, 1), N(0, 1)]' = [0.3N(0, 1), 0.3N(0, 1)]'$, where $N(0, 1)$ is a draw from a unit normal distribution.

The simulated data from the VAR uses starting values of $x_1 = x_2 = \alpha$, and a total simulation length of 160 is generated. Following Brüggemann (2004), an initial group of observations (120 in the present exercise) are discarded to minimize any influence of the starting values, and the last 10 observations are retained for evaluating the out-of-sample forecasting performance. That leaves a sample size of $T = 30$ for the estimation of the VAR.

The y values are obtained by subtracting the sample mean from the data

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[8]Having identical innovation contributions from both variables simplifies the calculation and evaluation of the forecast results.

$[x_1, \ldots, x_T]$,[9] and then the parameters Φ_1 and Φ_2 are estimated by OLS, i.e.:

$$\left[\hat{\Phi}_1, \hat{\Phi}_2\right] = \boldsymbol{y}_0 W' \left(WW'\right)^{-1} \tag{17}$$

where $\left[\hat{\Phi}_1, \hat{\Phi}_2\right]$ is the 2×8 (i.e. $N \times N^2 P$) matrix of estimated coefficients, $\boldsymbol{y}_0 = [y'_3, \ldots, y'_T]'$, and $W = \left[\boldsymbol{y}'_{-1}, \boldsymbol{y}'_{-2}\right]'$ with $\boldsymbol{y}_{-1} = \left[y'_2, \ldots, y'_{T-1}\right]'$ and $\boldsymbol{y}_{-2} = \left[y'_1, \ldots, y'_{T-2}\right]'$.

Typically the estimates of $\hat{\Phi}_1$ and $\hat{\Phi}_2$ would be used directly in the original VAR to obtain $\mathbb{E}_t[y_{t+\tau}]$, as indicated in the schematic diagram of equation 20. However, it is worthwhile for this illustration to proceed via the creation of the estimated $\hat{F}$ matrix in equation 2.[10] Forecasts for $Y_{t+\tau}$ at any horizon τ from Y_t may be obtained by:

$$\mathbb{E}_t[Y_{t+\tau}] = \hat{F}^\tau Y_t \tag{18}$$

and the forecasts of y_t $\mathbb{E}_t[y_{t+\tau}]$ are recovered from $\mathbb{E}_t[Y_{t+\tau}]$; $\mathbb{E}_t[y_{t+\tau}] = J\hat{F}^\tau Y_t$, where $J = [I_2, 0_2]$. Note that the identical forecasts could also be generated via the associated EVAR, which exploits the diagonal structure of $\hat{D}$ for the forecast of Z_t and then transforms the result back to Y_t via $\hat{V}$. That is, the associated EVAR is calculated by applying the eigensystem decomposition noted in equation 7, i.e. $\hat{F} = \hat{V}\hat{D}\hat{V}^{-1}$ which gives the EVAR $Z_t = \hat{D}Z_{t-1} + \eta_t$. Each individual CAR(1) from the EVAR gives $\mathbb{E}_t[Z_{j,t+\tau}] = \hat{D}_j^\tau Z_{j,t}$, and then for the system $\hat{V}\hat{D}^\tau Z_t = \hat{V}\hat{D}^\tau\hat{V}^{-1}Y_t = \left(\hat{V}\hat{D}\hat{V}^{-1}\right)^\tau Y_t = \hat{F}^\tau Y_t = \mathbb{E}_t[Y_{t+\tau}]$.

For the purposes of this illustration, the subset EVAR is generated by restricting to zero the coefficients in the column containing the smallest absolute value eigenvalue if that restriction is found to be appropriate. Specifically, a test of $H_0 : [0, \ldots, 0, D_{NP}]'_{NP} = 0$ against $H_1 : [0, \ldots, 0, D_{NP}]'_{NP} \neq 0$ is undertaken using an F-test with a right-tailed significance level of 0.05. The F-statistic is analogous to its OLS counterpart and is calculated as:

$$\text{F-stat} = \frac{(SSAR_{\text{R}} - SSAR_{\text{UR}})/NP}{SSAR_{\text{UR}}/[(T-NP)NP]}$$

where $SSAR_{\text{UR}} = \text{trace}\left(\eta_0\eta_0^\dagger\right)$ is the sum of squared absolute residuals from

[9]The mean $\hat{\mu}$ could also be estimated in conjunction with the VAR parameters $\hat{\Phi}_1$ and $\hat{\Phi}_2$ via $\hat{\alpha}$. However, Lütkepohl (2007) pp. 82-85 notes that the estimators $\bar{\boldsymbol{x}}$ and $\hat{\mu}$ are asymptotically equivalent. In any case, this adjustment to the data occurs prior the VAR and EVAR estimation and so won't influence the results.

[10]Alternatively, using the companion form data, $\hat{F}$ could be calculated directly as $\hat{F} = Y_0 Y'_{-1} \left(Y_{-1} Y'_{-1}\right)$, where $Y_0 = [Y_2, \ldots, Y_T]$ and $Y_{-1} = [Y_1, \ldots, Y_{T-1}]$. However, that requires more computation to obtain results identical to the constructed $\hat{F}$ matrix.

the unrestricted complex least squares (CLS) regression $Z_0 = \hat{D}Z_{-1} + \eta_0$,[11] and $SSR_{\mathrm{R}} = \mathrm{trace}\left(\eta_0^R \eta_0^{R\dagger}\right)$ is the sum of squared absolute residuals from the restricted CLS regression $Z_0 = \hat{D}^Q Z_{-1}^R + \eta_0^R$ where $Z_0 = \left[Z_{P+1}', \ldots, Z_T'\right]'$, $Z_{-1}^R = \left[Z_1', \ldots, Z_{NP-1}'\right]'$, and $\hat{D}^Q = Z_0 \left(Z_{-1}^R\right)^{\dagger} \left[Z_{-1}^R \left(Z_{-1}^R\right)^{\dagger}\right]^{-1}$. The numerator degrees of freedom is the number of restrictions $NP = 4$ and the denominator degrees of freedom is the number of observations TNP less the number of regressors $(NP)^2$; i.e. $(T - NP)\,NP = 96$.

Note that Z_{-1}^R is an $(NP - 1) \times (T - P)$ matrix so $\hat{D}^Q$ is an $NP \times (NP - 1)$ matrix of coefficients. If the zero restriction is not rejected, then concatenating a column of zeros onto $\hat{D}^Q$ gives the $NP \times NP$ matrix $\hat{D}^R$ with the explicit zero restrictions relative to $\hat{D}$. The restricted estimate of the $\hat{F}$ matrix is then calculated as $\hat{F}^R = \hat{V}\hat{D}^R\hat{V}^{-1}$, and $\hat{F}^R$ is used to obtain the EVAR forecasts.[12] Note that if the D_{NP} coefficient restricted to zero was complex, then the remaining complex conjugate coefficient will result in forecasts composed of complex numbers. To allow for this case (and with no impact on the case where the D_{NP} coefficient is real), the following is used as the EVAR forecast:

$$\mathbb{E}_t\left[Y_{t+\tau}\right] = \mathrm{Re}\left[\left(\hat{F}^R\right)^{\tau} Y_t\right] \tag{19}$$

The following is a schematic overview of the process described above, and below that is a numerical example of the results from applying the EVAR process to one of the simulations.

$$
\begin{array}{ccccc}
\left[\hat{\Phi}_1, \ldots, \hat{\Phi}_P\right], y_t & \Longrightarrow & \hat{F}, Y_t & \Longrightarrow & \hat{V}, \hat{D}, Z_t \\
\text{VAR} \Downarrow \text{path} & & & & \text{EVAR} \Downarrow \text{path} \\
\mathbb{E}_t\left[y_{t+\tau}\right] & \Longleftarrow & \hat{F}^R, Y_t & \Longleftarrow & \hat{V}, \hat{D}^R, Z_t
\end{array} \tag{20}
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[11]The trace($\cdot$) expression obtains by re-writing the EVAR in its seemingly-unrelated regression (SUR) form, i.e.:
$$
\begin{bmatrix} Z_{1,0}' \\ Z_{2,0}' \\ \vdots \\ Z_{NP,0}' \end{bmatrix} = \begin{bmatrix} Z_{-1} & 0 & \cdots & 0 \\ 0 & Z_{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & Z_{-1} \end{bmatrix} \begin{bmatrix} D_1^S \\ D_2^S \\ \vdots \\ D_{NP}^S \end{bmatrix} + \begin{bmatrix} \eta_{1,0}' \\ \eta_{2,0}' \\ \vdots \\ \eta_{NP,0}' \end{bmatrix}
$$
where the regressand is a $TNP \times 1$ vector, the regressors are a $TNP \times (NP)^2$ matrix, the coefficients are a $(NP)^2 \times 1$ vector of the vectors $D_1^S = [D_1, 0, \ldots, 0]'$, $D_2^S = [0, D_2, 0, \ldots, 0]'$, $\ldots$, $D_{NP}^S = [0, \ldots, 0, D_{NP}]'$, and the residuals are a $TNP \times 1$ vector. The restricted CLS regression is the same form with the last NP regressor columns deleted.

[12]Again, the forecasts could be obtained from the EVAR and transformed back to the companion form, but it is worthwhile for this illustration to calculate $\hat{F}^R$ explicitly.

$$
\hat{F} = \begin{bmatrix} 0.47 & -0.02 & -0.05 & 0.06 \\ 0.38 & 0.69 & 0.31 & -0.22 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{21a}
$$

$$
\hat{D} = \begin{bmatrix} 0.64 & 0 & 0 & 0 \\ 0 & 0.28 + 0.46i & 0 & 0 \\ 0 & 0 & 0.28 - 0.46i & 0 \\ 0 & 0 & 0 & -0.05 \end{bmatrix} \tag{21b}
$$

$$
\hat{D}^R = \begin{bmatrix} 0.64 & 0 & 0 & 0 \\ 0 & 0.28 + 0.46i & 0 & 0 \\ 0 & 0 & 0.28 - 0.46i & 0 \\ -0.01 & -0.01 + 0.01i & -0.01 - 0.01i & 0 \end{bmatrix} \tag{21c}
$$

$$
\hat{F}^R = \begin{bmatrix} 0.47 & -0.02 & -0.05 & 0.06 \\ 0.38 & 0.69 & 0.31 & -0.22 \\ 1.00 & -0.03 & 0.02 & 0.02 \\ -0.00 & 0.96 & 0.03 & 0.03 \end{bmatrix} \tag{21d}
$$

As an aside, note that the $\hat{F}^R$ matrix no longer contains the lower-left-hand identity matrix that relates the values of y_{t-1} in Y_t identically to the values of y_{t-1} in Y_{t-1}. The interpretation of this result is that $\hat{F}^R$ effectively provides alternative values of y_{t-1} for forecasting. Also note that because the changes to $\hat{F}$ are not limited to $\hat{\Phi}_1$ and $\hat{\Phi}_2$, then $\hat{F}^R$ does not strictly represent a subset VAR (even though it was obtained via a subset of the EVAR associated with the original VAR).[13]

Simulations of the process described above were undertaken 100,000 times. Figure 1 shows a histogram of the results of the individual squared forecast errors (SFEs) $(\mathbb{E}_t [y_{t+\tau}] - y_{t+\tau})'_k (\mathbb{E}_t [y_{t+\tau}] - y_{t+\tau})_k$ for the VAR method less the SFEs for the EVAR method at the horizon $\tau = 1$. The apparent skew toward negative values indicates that the EVAR method produces more instances of superior forecasts than the VAR method.

Figure 2 shows the results of the mean SFE (MSFE) by horizon τ:

$$
\text{MSFE}(\tau) = \frac{1}{K} \sum_{k=1}^{K} (\mathbb{E}_t [y_{t+\tau}] - y_{t+\tau})'_k (\mathbb{E}_t [y_{t+\tau}] - y_{t+\tau})_k \tag{22}
$$

---

[13]If desired, it should be possible to estimate the EVAR to restrict the smallest eigenvalue to zero while simultaneously retaining the traditional form of $\hat{F}$. However, that would come at the cost of forgoing the simplicity of the EVAR construction outlined in section 3.
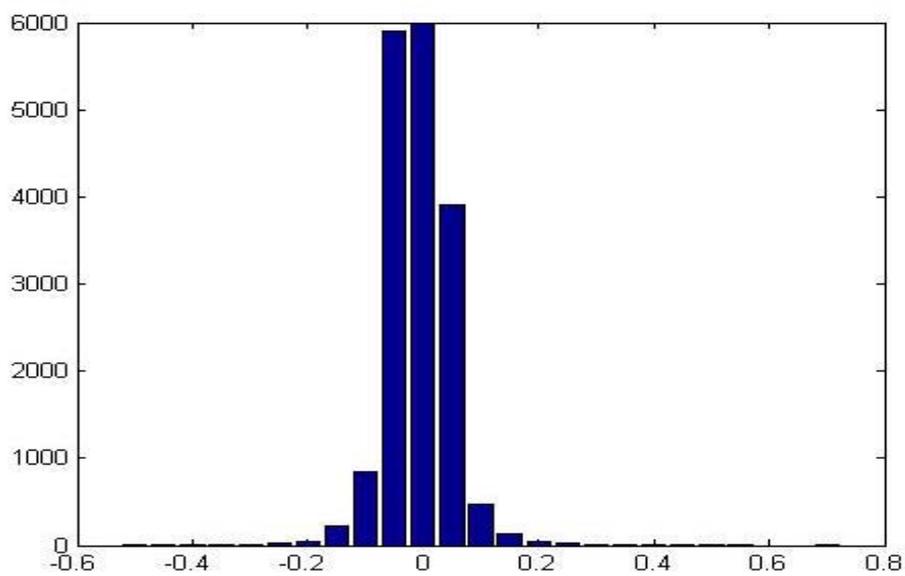
Figure 1: The distribution of squared VAR forecast errors less the squared EVAR forecast errors for the horizon $\tau = 5$. Note that the category of 0 has 88,311 observations, but has been truncated to show more detail in the other categories.
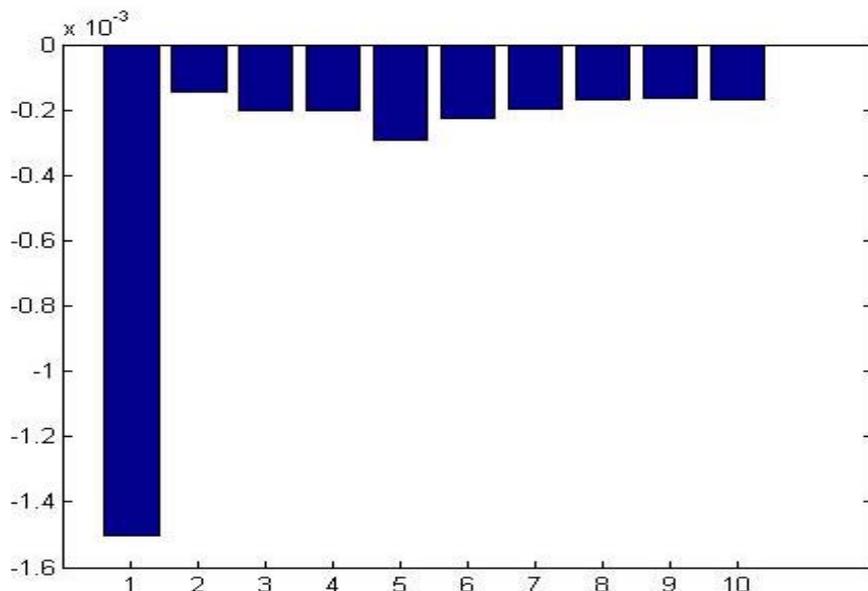
Figure 2: The mean of the squared VAR forecast errors less the squared EVAR forecast errors for horizons $\tau = 1$ to $\tau = 10$.

for the VAR method less the MSFE for the EVAR method. The results for all horizons are negative, indicating superior forecast performance on average for the EVAR method than the VAR method.

Of course, the results above are for a single VAR specification, and so offer no gauge of the sensitivity of the results for alternative VAR specifications. Indeed, one aspect that may bias the results towards the EVAR in this particular case is that the eigenvalues of the companion matrix F are, in order of absolute value, $\text{diag}(D) = [0.7693, 0.1154 + 0.1385i, 0.1154 - 0.1385i, 0]$. Hence, when the restriction of the smallest real eigenvalue to zero is made, the restricted EVAR may naturally have an advantage of being specified in the true form of the VAR, therefore giving less model bias. Further tests with alternative VARs will be undertaken in ongoing work.

The results above also use only one testing proceedure, and more sophisticated approaches may make better use of the information presented by the EVAR. For example, selecting the EVAR coefficient to potentially restrict to zero could be undertaken by considering both the magnitude of the coefficient and the associated residuals. Hence, rather than automatically testing the smallest coefficient for elimination, a larger EVAR coefficient may be tested if it is associated with relatively large residuals. Beyond testing a single re-

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striction, an obvious extension is to undertake sequence of tests to determine whether multiple groups of coefficients can be restricted to zero. In particular, the SUR form of the EVAR allows a higher degree of flexibility for testing zero restrictions than the EVAR in matrix form. Further investigation of these aspects also remain to be undertaken.

6 Potential application: bias correction for estimated VAR coefficients

It is well known that the least squares estimators for AR(1) models with a finite sample have a downward bias due, and so they underestimate the true persistence inherent in the univariate series. Examples of correcting for such biases are Andrews (1993) (correction calculated via numerical methods), Roy and Fuller (2001) (correction via an analytical approximation), Kim (2001) (correction via bootstrapping). Examples of testing those bias corrections methods include Kim (2003) and Falk and Roy (2005).

A similar estimation bias is known to exists in VAR models. For example, figure 3 illustrates the negatively skewed distribution from the VAR process used in section 5.2. However, examples of investigating those biases in finite-sample VAR models are less prevalent than for univariate AR models. The only two examples found were Abadir, Hadri and Tzavalis (1999) and Lawford and Stamatogiannis (2009).

Given the VAR model can be decomposed into a system of individual CAR(1) models, it seems possible in principle that a bias-correction applied to those individual components might offer an appropriate bias correction for the VAR system. As an illustration of this principle, the coefficient matrix in equation 21b has been adjusted by replacing the largest eigenvalue in the estimated coefficient matrix $\hat{D}$ (i.e. 0.64) with the Andrews (1993) median bias corrected value (i.e. 0.71), i.e.:

$$\hat{D}^S = \begin{bmatrix} 0.71 & 0 & 0 & 0 \\ 0 & 0.28 + 0.46i & 0 & 0 \\ 0 & 0 & 0.28 - 0.46i & 0 \\ 0 & 0 & 0 & -0.05 \end{bmatrix} \tag{23}$$

The matrix $\hat{D}^S$ is then used to calculate the alternative F matrix, i.e.:

$$\hat{F}^S = \begin{bmatrix} 0.50 & -0.01 & -0.04 & 0.07 \\ 0.48 & 0.71 & 0.31 & -0.22 \\ 1.05 & 0.01 & 0.00 & 0.00 \\ 0.15 & 1.04 & 0.01 & 0.00 \end{bmatrix} \tag{24}$$
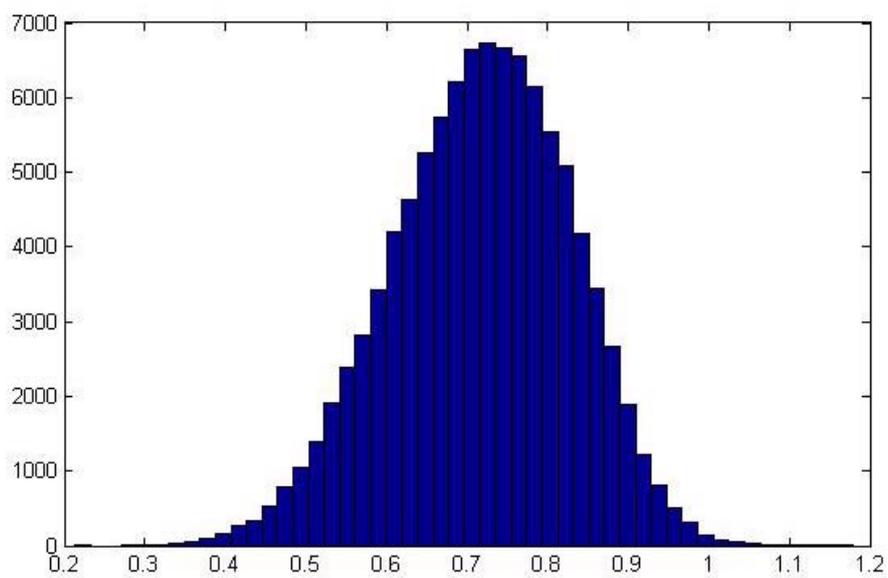
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Figure 3: The magnitude of the largest estimated eigenvalue from the process in section 5.2. The median is 0.7219, the mean is 0.7169, and the largest eigenvalue from the actual process is 0.7693. The variance, skewness, and kurtosis are 0.0135, -0.1584, and 2.7699 respectively.

An initial simulation based on this illustrative method did not produce superior MSFEs than the VAR method. Given its simplicity, that is not surprising. A robust method would likely need to account for all eigenvalues and the covariance of each CAR(1) as given by the residual matrix.

7   Conclusion

This article introduces the EVAR; a transformation of a standard VAR into a system of seemingly-unrelated univariate first-order complex autoregressions. The component structure of the EVAR is shown to be useful for interpreting the dynamics of the VAR, in terms of the underlying eigensystem and its innovations. The forecasting application successfully illustrates the potential for improving on VAR on forecasts by using an associated subset EVAR. The concept of correcting the biases of estimated VAR coefficients has been illustrated by numerical example. The applications of the EVAR in this article require further investigation from several perspectives.

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A   SUR form of the regression