Similar Tests for Mediation
Testing Hypotheses with Singularities

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1joint with Noud van Giersbergen and Hans van Ophem
Empirically Important
Empirically Important

Mediation models try to establish mechanisms of relations via other variables.
Empirically Important

1. Mediation models try to establish mechanisms of relations via other variables
2. Mediator Variable
Empirically Important

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2. Mediator Variable
   - Intermediary variable,
Empirically Important

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2. Mediator Variable
   - Intermediary variable,
   - Intervening variable
Introduction and Motivation: Mediation Effects

1 Empirically Important

1 Mediation models try to establish mechanisms of relations via other variables

2 Mediator Variable
   - Intermediary variable,
   - Intervening variable
   - Catalyst
Empirically Important

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Theoretically Interesting
Empirically Important

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Theoretically Interesting

- Non regular testing problem

\[ H_0 : \theta_1 \theta_2 = 0 \]
\[ H_1 : \theta_1 \theta_2 \neq 0 \]
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Theoretically Interesting

- Non regular testing problem

\[ H_0 : \theta_1 \theta_2 = 0 \]
\[ H_1 : \theta_1 \theta_2 \neq 0 \]

- Results from 1936 - today
Introduction: Testing for Mediation Effects: Empirically Important

- Baron and Kenny (1986)
Introduction: Testing for Mediation Effects: Empirically Important

- Baron and Kenny (1986)
Baron and Kenny (1986)

- 79,205 citations and ticking
Baron and Kenny (1986)

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• Baron and Kenny (1986)
  • The Moderator-Mediator Variable Distinction in Social Psychological Research: Conceptual, Strategic, and Statistical Considerations.
  • 79,205 citations and ticking
• Hayes (2018) 2nd ed. Introduction to Mediation, Moderation, and Conditional Process Analysis
• Psychology (positive and negative framing of questions)
Introduction: Testing for Mediation Effects: Empirically Important

- Baron and Kenny (1986)
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Psychology (positive and negative framing of questions)

Medicine

Economics
Baron and Kenny (1986)

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Psychology (positive and negative framing of questions)

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- e.g. Economic Stress $\Rightarrow$ Affect (state of mind) $\Rightarrow$ Withdraw as entrepreneur
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Psychology (positive and negative framing of questions)
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- e.g. Economic Stress ⇒ Affect (state of mind) ⇒ Withdraw as entrepreneur
Marketing
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Psychology (positive and negative framing of questions)

Medicine

Economics
- e.g. Economic Stress ⇒ Affect (state of mind) ⇒ Withdraw as entrepreneur

Marketing

Accountancy

Influence Channels (Econometric Game 2018)
Economic Stress $\Rightarrow$ Affect (state of mind) $\Rightarrow$ Withdraw as Entrepreneur
Mediation Hypothesis: 2 versions

\[ y = x_m \beta + x \delta + u \]
\[ x_m = x \alpha + \nu \]

- No mediation effect:
  \[ H_0 : \alpha \cdot \beta = 0 \]
Mediation Hypothesis: 2 versions

\[ y = x_m \beta + x \delta + u \]
\[ x_m = x \alpha + \nu \]

- No mediation effect:
  \[ H_0 : \alpha \cdot \beta = 0 \]

- Or no effect via \( x_m \)
  \[ y = x_m \beta + x \delta + u \]
  \[ y = x \delta_r + u_r \]

\[ H_0 : \delta = \delta_r \]
Mediation Hypothesis: 2 versions

\[ y = x_m \beta + x \delta + u \]
\[ x_m = x \alpha + \nu \]
\[ y = \delta_r x + u \]
Mediation Hypothesis: 2 versions

\[ y = x_m\beta + x\delta + u \]
\[ x_m = x\alpha + v \]
\[ y = \delta_r x + u \]

Comparing restricted \( \delta_r \) with unrestricted \( \delta \):

\[ \tilde{\delta}_r = (x'x)^{-1} x'y = (x'x)^{-1} x'(x_m\hat{\beta} + x\hat{\delta} + \hat{u}) \]
\[ = \hat{\delta} + (x'x)^{-1} x'x_m \hat{\beta} + 0 \]
\[ = \hat{\delta} + \hat{\alpha} \cdot \hat{\beta} \]

\[ \tilde{\delta}_r - \hat{\delta} = \hat{\alpha} \cdot \hat{\beta} \]
Mediation Hypothesis: 2 versions

\[ y = x_m \beta + x \delta + u \]
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\[ y = \delta_r x + u \]

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\[ \tilde{\delta}_r = (x'x)^{-1} x'y = (x'x)^{-1} x' (x_m \hat{\beta} + x \hat{\delta} + \hat{u}) \]
\[ = \hat{\delta} + (x'x)^{-1} x'x_m \hat{\beta} + 0 \]
\[ = \hat{\delta} + \hat{\alpha} \cdot \hat{\beta} \]

\[ \tilde{\delta}_r - \hat{\delta} = \hat{\alpha} \cdot \hat{\beta} \]

Or in model

\[ y = \beta x_m + \delta x + u = \beta (\alpha x + \nu) + \delta x + u = (\alpha \beta + \delta) x + (u + \beta \nu) \]

\[ H_0 : \delta_r = \delta \iff H_0 : \alpha \cdot \beta = 0 \]
Standard methods

$\alpha \cdot \beta$ based
Standard methods

1. $\alpha \cdot \beta$ based
   - Sobel (1982) $1^{st}$ order solution
Standard methods

1. $\alpha \cdot \beta$ based
   - Sobel (1982) $1^{st}$ order solution
   - Aroian (1947) $2^{nd}$ order exact solution

Validity of expansions depend $\alpha$ or $\beta$ when not zero.
Standard methods

1. $\alpha \cdot \beta$ based

- Sobel (1982) 1st order solution
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- Goodman (1960) Unbiased variance
- MacKinnon et al. (1998) $\alpha \beta / \sigma_{\alpha \beta}$
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   - MacKinnon and Lockwood (2001) asymmetric distribution
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2. **Product of standard normals & product of student-$t$’s, $z \ast z$, $t \ast t$**
   (Craig 1936, Drton 2009)
Standard methods

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   - Sobel (1982) 1\textsuperscript{st} order solution
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   (Craig 1936, Drton 2009)

3. Distribution depends on both $\alpha$ and $\beta$ (this is true also for LR and LM)
**Standard methods**

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3. **Distribution depends** on both $\alpha$ and $\beta$ (this is true also for LR and LM)

4. **Bootstrap**. Does not solve the problem. CV depend on $\alpha$ and $\beta$
Standard methods

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   - Sobel (1982) 1st order solution
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5. Validity of expansions depend $\alpha$ or $\beta$ when not zero.
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   - Sobel (1982) 1\(^{st}\) order solution
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6. Implementation R-package
Estimation: Maximum Likelihood

- \( f(y, x_m | x) = f(y | x_m, x) f(x_m | x) \)

\[
\ell = - \frac{1}{2\sigma_{11}} \sum_{i=1}^{n} (y_i - \beta x_{m,i} - \delta x_i)^2 - \frac{1}{2\sigma_{22}} \sum_{i=1}^{n} (x_{m,i} - \alpha x_i)^2 - \frac{n}{2} \log(\sigma_{11}\sigma_{22})
\]
Estimation: Maximum Likelihood

- \( f(y, x_m | x) = f(y | x_m, x) f(x_m | x) \)

\[
\ell = -\frac{1}{2\sigma_{11}} \sum_{i=1}^{n} (y_i - \beta x_m,i - \delta x_i)^2 - \frac{1}{2\sigma_{22}} \sum_{i=1}^{n} (x_m,i - \alpha x_i)^2 - \frac{n}{2} \log (\sigma_{11} \sigma_{22})
\]

- **Expected Fisher Information** block diagonal

\[
\begin{pmatrix}
\frac{x'x}{\sigma_{11}} + \frac{N \sigma_{22}}{\sigma_{11}} & \frac{x'x}{\sigma_{11}} & 0 & 0 & 0 \\
\frac{x'x}{\sigma_{11}} & \frac{x'x}{\sigma_{11}} & 0 & 0 & 0 \\
0 & 0 & \frac{N}{2\sigma_{11}^2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sigma_{22}} x'x & 0 \\
0 & 0 & 0 & 0 & \frac{N}{2\sigma_{22}^2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\beta \\
\delta \\
\sigma_{11} \\
\alpha \\
\sigma_{22}
\end{pmatrix}
\]
Estimation: Maximum Likelihood

- \( f(y, x_m | x) = f(y | x_m, x) f(x_m | x) \)

- **Expected Fisher Information** block diagonal

- **Observed Information Matrix** also block diagonal

\[
\begin{pmatrix}
\frac{x'x}{\sigma_{11}} + N \frac{\sigma_{22}}{\sigma_{11}} & \frac{x'x}{\sigma_{11}} & 0 & 0 & 0 \\
\frac{x'x}{\sigma_{11}} & \frac{x'x}{\sigma_{11}} & 0 & 0 & 0 \\
0 & 0 & N \frac{1}{2\sigma_{11}^2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sigma_{22}} x'x & 0 \\
0 & 0 & 0 & 0 & \frac{N}{2\sigma_{22}^2}
\end{pmatrix}
\begin{pmatrix}
\beta \\
\delta \\
\sigma_{11} \\
\alpha \\
\sigma_{22}
\end{pmatrix}
\]
Estimation: Maximum Likelihood

- \( f(y, x_m | x) = f(y | x_m, x) f(x_m | x) \)

\[
\ell = -\frac{1}{2\sigma_{11}} \sum_{i=1}^{n} (y_i - \beta x_{m,i} - \delta x_i)^2 - \frac{1}{2\sigma_{22}} \sum_{i=1}^{n} (x_{m,i} - \alpha x_i)^2 - \frac{n}{2} \log(\sigma_{11}\sigma_{22})
\]

- **Expected Fisher Information** block diagonal

\[
\begin{pmatrix}
\frac{x'x}{\sigma_{11}} + N \frac{\sigma_{22}}{\sigma_{11}} & \frac{x'x}{\sigma_{11}} & 0 & 0 & 0 \\
\frac{x'x}{\sigma_{11}} & \frac{x'x}{\sigma_{11}} & 0 & 0 & 0 \\
0 & 0 & N \frac{\sigma_{22}}{2\sigma_{11}^2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sigma_{22}} x'x & 0 \\
0 & 0 & 0 & 0 & N \frac{\sigma_{22}}{2\sigma_{22}^2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\beta \\
\delta \\
\sigma_{11} \\
\alpha \\
\sigma_{22}
\end{pmatrix}
\]

- **Observed Information Matrix** also block diagonal

- Asymptotically **Normal** and **Independent**: \( \hat{\beta}, \hat{\alpha}, \hat{\sigma}_{11}, \hat{\sigma}_{22} \)
\[ \sqrt{N} \left( \begin{array}{c} \hat{\beta} - \beta, \\ \hat{\alpha} - \alpha \\ \hat{\sigma}_{11} - \sigma_{11} \\ \hat{\sigma}_{22} - \sigma_{22} \end{array} \right) \xrightarrow{d} N(0, V) \]

- We use approximation:
  \[ \left( \begin{array}{c} t_{\alpha} \\ t_{\beta} \end{array} \right) \xrightarrow{d} N \]
\[
\sqrt{N} \left( \begin{array}{c}
\hat{\beta} - \beta, \\
\hat{\alpha} - \alpha \\
\hat{\sigma}_{11} - \sigma_{11} \\
\hat{\sigma}_{22} - \sigma_{22}
\end{array} \right) \xrightarrow{d} N(0, V)
\]

- We use approximation:
\[
\left( \begin{array}{c}
t_\alpha \\
t_\beta
\end{array} \right) \xrightarrow{d} N
\]

- Throughout we use:
\[
\left( \begin{array}{c}
t_\alpha - \alpha / \sigma_\alpha \\
t_\beta - \beta / \sigma_\beta
\end{array} \right) \xrightarrow{d} N(0, I_2)
\]
Wald: $H_0 : r(\alpha, \beta) = \alpha \cdot \beta = 0$
Wald: $H_0 : r(\alpha, \beta) = \alpha \cdot \beta = 0$

$W = r' (R' VR)^{-1} r \xrightarrow{d} \chi^2_1$
Testing

- **Wald**: $H_0: r(\alpha, \beta) = \alpha \cdot \beta = 0$
- $W = r' (R' VR)^{-1} r \xrightarrow{d} \chi^2_1$

$$R(\alpha, \beta) = \frac{\partial \alpha \cdot \beta}{\partial (\alpha, \beta)'} = \left( \begin{array}{c} \beta \\ \alpha \end{array} \right)$$
Testing

- **Wald:** $H_0 : r(\alpha, \beta) = \alpha \cdot \beta = 0$
- $W = r'(R'VR)^{-1}r \xrightarrow{d} \chi^2_1$

\[
R(\alpha, \beta) = \frac{\partial \alpha \cdot \beta}{\partial (\alpha, \beta)'} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}
\]

\[
W = ab \left( (b \ a)' \begin{pmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \right)^{-1} ab
\]

\[
= \frac{a^2 b^2}{a^2 \sigma_b^2 + b^2 \sigma_a^2} \cdot \left( \frac{\sigma^2_a \sigma^2_b}{\sigma^2_a \sigma^2_b} \right)^{-1}
\]

\[
= \frac{t^2_\alpha t^2_\beta}{t^2_\alpha + t^2_\beta}
\]
**Wald:** \( H_0 : r(\alpha, \beta) = \alpha \cdot \beta = 0 \)

\( W = r'(R'VR)^{-1} r \overset{d}{\rightarrow} \chi^2_1 \)

\[
R(\alpha, \beta) = \frac{\partial \alpha \cdot \beta}{\partial (\alpha, \beta)'} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}
\]

\[
W = ab \begin{pmatrix} b & a \end{pmatrix}' \begin{pmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}^{-1} ab
= \frac{a^2 b^2}{a^2 \sigma_b^2 + b^2 \sigma_a^2} \cdot \frac{(\sigma_a^2 \sigma_b^2)^{-1}}{(\sigma_a^2 \sigma_b^2)^{-1}}
= \frac{t_\alpha^2 t_\beta^2}{t_\alpha^2 + t_\beta^2}
\]

\( R'VR = 0 \) if \( \beta = 0 \) and \( \alpha = 0 \)


- **Wald:**

\[
W = \frac{t_1^2 t_2^2}{t_1^2 + t_2^2} \xrightarrow{d} \begin{cases} 
\chi_1^2 : \alpha = 0 \text{ or } \beta = 0 \text{ but not both} \\
\frac{1}{4} \chi_1^2 : \alpha = \beta = 0
\end{cases}
\]
Wald:

\[ W = \frac{t_1^2 t_2^2}{t_1^2 + t_2^2} \xrightarrow{d} \left\{ \begin{array}{l} \chi_1^2 : \alpha = 0 \text{ or } \beta = 0 \text{ but not both} \\
\frac{1}{4} \chi_1^2 : \alpha = \beta = 0 \end{array} \right. \]

LR: Van Giersbergen (2018)

\[ LR = \min \{ |t_1|, |t_2| \} \]
Testing

- **Wald:**

\[ W = \frac{t_1^2 t_2^2}{t_1^2 + t_2^2} \overset{d}{\rightarrow} \begin{cases} \chi_1^2 : \alpha = 0 \text{ or } \beta = 0 \text{ but not both} \\ \frac{1}{4}\chi_1^2 : \alpha = \beta = 0 \end{cases} \]

- **LR:** Van Giersbergen (2018)

\[ LR = \min \{|t_1|, |t_2|\} \]

- Reject when both \( H_0\alpha : \alpha = 0 \) and \( H_0\beta : \beta = 0 \) are rejected:

\[ P[G|\alpha, \beta] = P[A \cap B|\alpha, \beta] = P[A|\alpha] \cdot P[B|\beta] = \begin{cases} 0.05^2 = 0.0025 \text{ if } \alpha = 0 \land \beta = 0 \\ 0.05 \text{ if } \alpha \rightarrow \infty \text{ or } \beta \rightarrow \infty \end{cases} \]
Wald:

\[ W = \frac{t_1^2 t_2^2}{t_1^2 + t_2^2} \xrightarrow{d} \begin{cases} \chi_1^2 : \alpha = 0 \text{ or } \beta = 0 \text{ but not both} \\ \frac{1}{4}\chi_1^2 : \alpha = \beta = 0 \end{cases} \]

LR: Van Giersbergen (2018)

\[ LR = \min \{ |t_1|, |t_2| \} \]

Reject when both \( H_{0\alpha} : \alpha = 0 \) and \( H_{0\beta} : \beta = 0 \) are rejected:

\[ P[G|\alpha, \beta] = P[A \cap B|\alpha, \beta] = P[A|\alpha] \cdot P[B|\beta] \]

\[ = \begin{cases} 0.05^2 = 0.0025 \text{ if } \alpha = 0 \land \beta = 0 \\ 0.05 \text{ if } \alpha \to \infty \text{ or } \beta \to \infty \end{cases} \]

depend on \( \alpha \) and \( \beta \)
Testing

- **Wald:**

  \[ W = \frac{t_1^2 t_2^2}{t_1^2 + t_2^2} \xrightarrow{d} \begin{cases} 
  \chi_1^2 : \alpha = 0 \text{ or } \beta = 0 \text{ but not both} \\
  \frac{1}{4} \chi_1^2 : \alpha = \beta = 0 
  \end{cases} \]

- **LR:** Van Giersbergen (2018)

  \[ LR = \min \{|t_1|, |t_2|\} \]

  - Reject when both \( H_0\alpha : \alpha = 0 \) and \( H_0\beta : \beta = 0 \) are rejected:

    \[ P[G|\alpha, \beta] = P[A \cap B|\alpha, \beta] = P[A|\alpha] \cdot P[B|\beta] \]

    \[ = \begin{cases} 
    0.05^2 = 0.0025 & \text{if } \alpha = 0 \land \beta = 0 \\
    0.05 & \text{if } \alpha \to \infty \text{ or } \beta \to \infty 
    \end{cases} \]

    - depend on \( \alpha \) and \( \beta \)

- **LM Score:** Distribution also depends on true value of \( \alpha \) and \( \beta \).
Testing: LM test

- Score test: 3 version  
  \[ H_{0\alpha} : \alpha = 0 \quad H_{0\beta} : \beta = 0 \quad H_{0\alpha\beta} : \alpha = 0 \land \beta = 0 \]

\[
s (\tilde{\theta}_{\alpha=0}) = \begin{pmatrix} 0 \\ 0 \\ \tilde{v}' \tilde{x} / N \end{pmatrix}; \quad s (\tilde{\theta}_{\beta=0}) = \begin{pmatrix} \tilde{u}' x_m / N \\ 0 \\ 0 \end{pmatrix}; \quad s (\tilde{\theta}_{\alpha=0 \land \beta=0}) = \begin{pmatrix} \tilde{u}' x_m / N \\ \tilde{u}' \tilde{u} / N \\ 0 \end{pmatrix}
\]
Testing: LM test

- Score test: 3 version
  - $H_0^\alpha : \alpha = 0$  
  - $H_0^\beta : \beta = 0$  
  - $H_0^{\alpha\beta} : \alpha = 0 \land \beta = 0$

$$s(\tilde{\theta}_{\alpha=0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{v}'x \tilde{v}'/N \end{pmatrix}; 
 s(\tilde{\theta}_{\beta=0}) = \begin{pmatrix} \tilde{u}'x_m \\ \tilde{u}'\tilde{u}/N \\ 0 \\ 0 \end{pmatrix}; 
 s(\tilde{\theta}_{\alpha=0\land\beta=0}) = \begin{pmatrix} \tilde{u}'x_m \\ \tilde{u}'\tilde{u}/N \\ 0 \\ 0 \end{pmatrix};$$

- $LM = s(\tilde{\theta})' I_{\tilde{\theta}}^{-1} s(\tilde{\theta})$

$$LM_{\alpha=0} = N \frac{x'_m x'_m}{x'x x'_m x_m}$$

$$LM_{\beta=0} = N \frac{\tilde{u}'x_m x'_m \tilde{u}}{\tilde{u}'\tilde{u} \tilde{v}'\tilde{v}}$$

$$LM_{\alpha=0\land\beta=0} = N \frac{\tilde{u}'x_m x'_m \tilde{u}}{\tilde{u}'\tilde{u} \tilde{v}'\tilde{v}} + N \frac{x'_m x'_m}{x'x x'_m x_m}$$
Testing: LM test

- Score test: 3 version  
  \(H_0\alpha : \alpha = 0\)  \(H_0\beta : \beta = 0\)  \(H_0\alpha\beta : \alpha = 0 \land \beta = 0\)

\[
\begin{align*}
\mathbf{s} (\tilde{\theta}_{\alpha=0}) &= \begin{pmatrix}
0 \\
0 \\
0 \\
\tilde{\nu}'x \\
\tilde{\nu}'\tilde{v}/N
\end{pmatrix} ;
\mathbf{s} (\tilde{\theta}_{\beta=0}) &= \begin{pmatrix}
\tilde{u}'x_m \\
0 \\
0 \\
0
\end{pmatrix} ;
\mathbf{s} (\tilde{\theta}_{\alpha=0\land\beta=0}) &= \begin{pmatrix}
\tilde{u}'x_m \\
\tilde{u}'\tilde{u}/N \\
0 \\
0
\end{pmatrix}
\end{align*}
\]

- \(LM = \mathbf{s} (\tilde{\theta})' \mathbf{I}_{\tilde{\theta}}^{-1} \mathbf{s} (\tilde{\theta})\)

\[
\begin{align*}
LM_{\alpha=0} &= N \frac{x_m'x_m}{x'x x_m'x_m} \\
LM_{\beta=0} &= N \frac{\tilde{u}'x_mx_m\tilde{u}}{\tilde{u}'\tilde{u} \tilde{v}'\tilde{v}} \\
LM_{\alpha=0\land\beta=0} &= N \frac{\tilde{u}'x_mx_m\tilde{u}}{\tilde{u}'\tilde{u} \tilde{v}'\tilde{v}} + N \frac{x_m'x_m}{x'x x_m'x_m}
\end{align*}
\]

- Squared t-statistics but using restricted variance estimates
Distributions depend on $\alpha$ and $\beta$

$\Rightarrow$ Size depends on $\alpha$ or $\beta$
Distributions depend on $\alpha$ and $\beta$

- $\Rightarrow$ Size depends on $\alpha$ or $\beta$
- Wald
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Distributions depend on $\alpha$ and $\beta$

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  - ...
  - Dufour, Renault, Zinde-Walsh (2017)
\[ H_0 : \theta \in \Theta_0 \quad \text{vs} \quad H_1 : \theta \in \Theta \setminus \Theta_0 \]

Let \( \omega \) be the boundary between \( H_0 \) and \( H_1 \).
\[ H_0 : \theta \in \Theta_0 \ \text{vs} \ \ H_1 : \theta \in \Theta \setminus \Theta_0 \]

Let \( \omega \) be the boundary between \( H_0 \) and \( H_1 \).

**Definition**

A test is called similar on the boundary \( \omega \) if

\[ P [\text{reject}] = \text{const} \text{ for all } \theta \in \omega \]
\[ H_0 : \theta \in \Theta_0 \ vs \ H_1 : \theta \in \Theta \setminus \Theta_0 \]

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- For Different approach:
  
  **Note:**
\[ H_0 : \theta \in \Theta_0 \text{ vs } H_1 : \theta \in \Theta \setminus \Theta_0 \]

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- For Different approach:
  **Note:**

1. W, LR, LM all depend on \( t \) statistics
Let $\omega$ be the boundary between $H_0$ and $H_1$.

**Definition**

A test is called similar on the boundary $\omega$ if

$$P[\text{reject}] = \text{const} \text{ for all } \theta \in \omega$$

For Different approach:

**Note:**

1. $W$, LR, LM all depend on $t$ statistics
2. $t$ carries the 2 dimensional relevant information for testing $H_0$
Critical Region

Definition

**CR: Critical Region** reject $H_0$ if

$$ (t_\alpha, t_\beta) \in CR \subset \mathbb{R}^2 $$

- Acceptance region $AR = \overline{CR}$
**Critical Region**

**Definition**

**CR: Critical Region**

reject $H_0$ if

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  - LR CR
Definition

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**CR: Critical Region** reject $H_0$ if

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- Acceptance region $AR = \overline{CR}$
- A test statistic + critical value defines a critical region
  - Wald CR
  - LR CR
  - LM CR
- We seek solution directly in terms of CR
Similar Tests for Mediation
Similar Tests for Mediation
Problem is symmetric in

\[ \alpha \leftrightarrow \beta \text{ (after standardization)} \]

\[ \alpha \leftrightarrow -\alpha \text{ or } \beta \leftrightarrow -\beta \]

Critical region should reflect this:

- Define 1/8 of CR in one octant: North-East to East
- Remaining 7 parts follow by reflections
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Critical Region Boundary

Definition

\[ g: \mathbb{R} \rightarrow \mathbb{R} \] defines the Critical Region \( \text{CR}_g = \{(t_\alpha, t_\beta) \in \mathbb{R}^2 | |t_\alpha| > g(t_\beta) \cap |t_\beta| > g(t_\alpha)\} \]

Acceptance Region \( \text{AR}_g = \{(t_\alpha, t_\beta) \in \mathbb{R}^2 | |t_\alpha| \leq g(t_\beta) \cup |t_\alpha| \leq g(t_\beta)\} \)

Definition \( g(\cdot) \) is said to be a similar boundary function if \( \text{Pr}[t \in \text{CR}_g | H_0] = 0.05 \) \( \forall (\alpha, \beta) \in \mathbb{R}^2 \) with \( \alpha \cdot \beta = 0. \)
Definition

**Boundary function.** $g : \mathbb{R} \rightarrow \mathbb{R}$ defines the Critical Region

\[
CR_g = \{ (t_\alpha, t_\beta) \in \mathbb{R}^2 | \quad |t_\alpha| > g(t_\beta) \cap |t_\beta| > g(t_\alpha) \}
\]

Acceptance Region

\[
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\[
AR_g = \{(t_\alpha, t_\beta) \in \mathbb{R}^2 | \ |t_\alpha| \leq g(t_\beta) \cup |t_\alpha| \leq g(t_\beta)\}
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Definition

**Boundary function.** $g : \mathbb{R} \to \mathbb{R}$ defines the Critical Region

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Acceptance Region

$$AR_g = \{(t_\alpha, t_\beta) \in \mathbb{R}^2 | \ |t_\alpha| \leq g(t_\beta) \cup |t_\alpha| \leq g(t_\beta)\}$$

Definition

$g (\cdot)$ is said to be a **similar boundary function** if

$$P \left[ t \in CR_g \mid H_0 \right] = 0.05 \quad \forall (\alpha, \beta) \in \mathbb{R}^2 \text{ with } \alpha \cdot \beta = 0.$$
Proposition (i) The LR boundary function is not similar.
(ii) The Wald boundary is not similar. (ii) The LM boundary is not similar.

Proof. (i) If $\alpha \rightarrow \infty$ then $P[\text{Reject}] = P[|t| > 1.96] = 0.05$.

If $\alpha = 0$ and $\beta = 0$ by independence of $t_\alpha$ and $t_\beta$,

$$P[|t| > 1.96] \cdot P[|t| > 1.96] = 0.0025 < 0.05$$

(ii) Drton 2009: distribution $t_\alpha t_\beta$ depends on $\alpha$ and $\beta$ under $H_0$.

Fixed critical value :: rejection probabilities vary with $\alpha$ and $\beta$. 

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Similar Tests for Mediation

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LR, LM and Wald not Similar

Proposition

(i) The LR boundary function is not similar.
(ii) The Wald boundary is not similar. (ii) The LM boundary is not similar.

Proof.

(i) $\alpha \to \infty$ then $P[\text{Reject}] = P[t_\beta > 1.96] = 0.05$

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Similar Tests for Mediation 

January 20, 2019 22 / 50
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(i) The LR boundary function is not similar.
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Proof.
(i) \(\alpha \to \infty\) then \(P[\text{Reject}] = P[t_{\beta} > 1.96] = 0.05\).
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If \( \alpha = 0 \) and \( \beta = 0 \) by independence of \( t_\alpha \) and \( t_\beta \)

\[
P [ |t_\alpha| > 1.96] \cdot P [ |t_\alpha| > 1.96] = 0.0025 << 0.05
\]

(ii) Drton 2009 : distribution \( t_\alpha \ t_\beta \) depends on \( \alpha \) and \( \beta \) under \( H_0 \).
fixed critical value :: rejection probabilities vary with \( \alpha \) and \( \beta \)
Non-Existence of Similar test

Theorem

No similar boundary function $g(\cdot)$ exists for testing $H_0: \alpha \cdot \beta = 0$.

Problem symmetric in $\alpha$ and $\beta$ \quad $\therefore H_0: \beta = 0$ and $\alpha \in \mathbb{R}$.

Proof.

[Proof]

Probability of not rejecting $H_0$ should equal 0.95 $\forall \alpha \in \mathbb{R}$

$P[T \in \text{AR} g | \alpha, g(\cdot)] = P[T \in \text{CR} g | H_0] = P[|T_{\alpha}| \leq |g(T_{\beta})| \cup |T_{\beta}| \leq |g(T_{\alpha})| \land \alpha \in \mathbb{R}]$

Under $H_0$: $\beta = 0$ and $\alpha \in \mathbb{R}$: $t$-statistics are independent normal $T_{\alpha} \sim N(\alpha \sigma_{\alpha}, 1)$ and $T_{\beta} \sim N(0, 1)$.
Non-Existence of Similar test

Theorem

No similar boundary function $g(\cdot)$ exists for testing $H_0 : \alpha \cdot \beta = 0$. 
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Non-Existence of Similar test

- Theorem

No similar boundary function $g(\cdot)$ exists for testing $H_0 : \alpha \cdot \beta = 0$.

- Problem symmetric in $\alpha$ and $\beta$: $H_0 : \beta = 0$ and $\alpha \in \mathbb{R}$.

Proof.

[Proof [AR version]] Probability of not rejecting $H_0$ should equal 0.95 $\forall \alpha \in \mathbb{R}$

$$P \left[ T \in AR_g | \alpha, g(\cdot) \right] = P \left[ T \in CR_g | H_0 \right]$$

$$= P \left[ \left| T_\alpha \right| \leq g(T_\beta) \cup \left| T_\beta \right| \leq g(T_\alpha) | \beta = 0 \wedge \alpha \in \mathbb{R} \right]$$

Under $H_0 : \beta = 0$ and $\alpha \in \mathbb{R}$: $t$ statistics are independent normal

$T_\alpha \sim N \left( \frac{\alpha}{\sigma_\alpha}, 1 \right)$ and $T_\beta \sim N (0, 1)$. 

Non-Existence of Similar test

Proof.

[Proof continued] \( P \left[ T \in AR_g | \alpha, g(\cdot) \right] \)

\[
= 2 \int_{-\infty}^{+\infty} f(t_\alpha | \alpha) \left[ \int_0^{g(t_\alpha)} f(t_\beta | 0) + \int_{g^{-1}(t_\alpha)}^{+\infty} f(t_\beta | 0) \right] dt_\beta dt_\alpha
\]

\[
= 2 \int_{-\infty}^{+\infty} \phi \left( t_\alpha - \frac{\alpha}{\sigma_\alpha} \right) \left[ \int_0^{g(t_\alpha)} \phi(t_\beta) + \int_{g^{-1}(t_\alpha)}^{+\infty} \phi(t_\beta) \right] dt_\beta dt_\alpha
\]

\[
= 2 \int_{-\infty}^{+\infty} \phi \left( t_\alpha - \frac{\alpha}{\sigma_\alpha} \right) \left[ \Phi(g(t_\alpha)) - \frac{1}{2} + 1 - \Phi(g^{-1}(t_\alpha)) \right] dt_\alpha
\]

\[
= 2 \int_{-\infty}^{+\infty} \phi \left( t - \frac{\alpha}{\sigma_\alpha} \right) \left[ \Phi(g(t_\alpha)) - \Phi(g^{-1}(t_\alpha)) + \frac{1}{2} \right] dt_\alpha
\]

\( \Rightarrow \) restriction on \( g(\cdot) \)
Non-Existence of Similar test

Proof.

\[ P \left[ T \in AR_g|\alpha, g(\cdot) \right] - 0.95 = 0 = \int_{-\infty}^{+\infty} \phi \left( t - \frac{\alpha}{\sigma_\alpha} \right) F(t) \, dt \quad \forall \alpha \in \mathbb{R} \]

with

\[ F(t) = 2 \cdot \left[ \Phi(g(t)) - \Phi(g^{-1}(t)) + \frac{1}{2} - 0.95/2 \right] \]

Normal distribution \( N(\mu, 1) \) is a one parameter full exponential family and therefore complete: \( F(T) = 0 \) is the only function with expectation 0 for all values of \( \mu \).

Hence \( g(t) \) must satisfy

\[ \Phi(g(t)) - \Phi(g^{-1}(t)) = -0.025 \]

\( g(0) = 0 \) implies \( g^{-1}(0) = 0 \) \( \square \)
Proof.

[Proof continued]  
$g(0) = 0$ implies $g^{-1}(0) = 0$. Hence  

$$
\Phi(g(t)) - \Phi(g^{-1}(t)) = \Phi(0) - \Phi(0) = 0 \neq -0.025
$$

a contradiction.  
No similar boundary function $g(t)$ exists.

Q.E.D.
Non-Existence of Similar test

Proof.

[Proof Extended] If we were to entertain the possibility $g(L) = 0$ such that $g^{-1}(0) = L$ and not defined for $-L < t < L$ (origin part of CR!)

\[
P[T \in AR_g | \alpha, g(\cdot)]
\]

\[
= 2 \int_{-\infty}^{L} \phi \left( t_\alpha - \frac{\alpha}{\sigma_\alpha} \right) \left[ \int_{0}^{g(t_\alpha)} \phi (t_\beta) + \int_{g^{-1}(t_\alpha)}^{+\infty} \phi (t_\beta) \right] dt_\beta dt_\alpha + \\
+ 2 \int_{+L}^{+\infty} \phi \left( t_\alpha - \frac{\alpha}{\sigma_\alpha} \right) \left[ \int_{0}^{g(t_\alpha)} \phi (t_\beta) + \int_{g^{-1}(t_\alpha)}^{+\infty} \phi (t_\beta) \right] dt_\beta dt_\alpha + \\
+ 2 \int_{-L}^{+L} \phi \left( t_\alpha - \frac{\alpha}{\sigma_\alpha} \right) \int_{g^{-1}(t_\alpha)}^{+\infty} \phi (t_\beta) dt_\beta dt_\alpha
\]

\[
P[T \in AR_g | \alpha, g(\cdot)] - 0.95 = \int_{-\infty}^{+\infty} \phi \left( t - \frac{\alpha}{\sigma_\alpha} \right) F(t) dt = 0
\]
Non-Existence of Similar test

Proof.

[Proof Extended] with $F(t)$

\[
\begin{align*}
2 & \left[ \int_0^{g(t)} \phi(t_\beta) + \int_{g^{-1}(t)}^{+\infty} \phi(t_\beta) \right] dt_\beta \cdot I_{(\infty,-L] \cup [L,\infty)}(t) + \\
+2 & \int_{g^{-1}(t)}^{+\infty} \phi(t_\beta) dt_\beta \cdot I_{(-L,L)}(t) - 0.95 \\
= & \left[ \Phi(g(t)) - 1/2 + 1 - \Phi(g^{-1}(t)) \right] \cdot I_{(-\infty,-L] \cup [L,\infty)}(t) + \\
+2 & \left( 1 - \Phi(g^{-1}(t)) \right) \cdot I_{(-L,L)}(t) - 0.95
\end{align*}
\]

and $I_A(t)$ the indicator function.

By the completeness of the normal distribution $F(t) = 0$. For $-L < t < L$ this implies that $2 \cdot (1 - \Phi(g^{-1}(t))) - 0.95 = 0$ so $g^{-1}(t) = \Phi^{-1}(1 - 0.475) = \Phi^{-1}(0.525) \approx 0.0627$, since only the last integral is non-zero in this interval.

This **contradicts** the premise that $g^{-1}(0) = L$. 

\[\square\]
Theory: No exact similar test exists (even when assuming exact joint normality of the t statistics)

Yes!

Very close to 5% ($<10^{-4}$)

Uniformly over $\alpha$ and $\beta$ under $H_0$
**Theory**: No exact similar test exists 
(even when assuming exact joint normality of the t statistics)

**Practice**: Can we do better than a Null Rej Prob of 0.25% ?
**Theory**: No exact similar test exists (even when assuming exact joint normality of the t statistics)

**Practice**: Can we do better than a Null Rej Prob of 0.25%?

Yes!
Non-Existence of Similar test: Theory and Practice

- **Theory**: No exact similar test exists (even when assuming exact joint normality of the t statistics)
- **Practice**: Can we do better than a Null Rej Prob of 0.25%?
  - Yes!
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Theory: No exact similar test exists
  (even when assuming exact joint normality of the t statistics)
Practice: Can we do better than a Null Rej Prob of 0.25%?
Yes!
Very close to 5% ($< 10^{-4}$)
Uniformly over $\alpha$ and $\beta$ under $H_0$
1 Regard $H_0 : \alpha \cdot \beta$ as $H_{0\alpha} : \alpha = 0$ OR $H_{0\beta} : \beta = 0$. 
Regard $H_0 : \alpha \cdot \beta$ as $H_{0\alpha} : \alpha = 0 \text{ OR } H_{0\beta} : \beta = 0$

1. $\hat{\alpha}$ and $\hat{\beta}$ $\Rightarrow$ $t_{\hat{\alpha}}$ and $t_{\hat{\beta}}$ (possibly noncentral) student t distributions with $n - 2$ and $n - 3$ dof.
1. **Regard** $H_0 : \alpha \cdot \beta$ as $H_{0\alpha} : \alpha = 0$ OR $H_{0\beta} : \beta = 0$

2. $\hat{\alpha}$ and $\hat{\beta} \Rightarrow t_{\hat{\alpha}}$ and $t_{\hat{\beta}}$ (possibly noncentral) student t distributions with $n - 2$ and $n - 3$ dof.

3. **Approximate** by $N(\mu, 1)$ and $N(0, 1)$

   $\hat{\alpha}$ and $\hat{\beta}$ are uncorrelated (but $\hat{\beta}$ is mixed Gaussian)
1. **Regard** \( H_0 : \alpha \cdot \beta \) as \( H_{0\alpha} : \alpha = 0 \text{ OR } H_{0\beta} : \beta = 0 \)

2. \( \hat{\alpha} \) and \( \hat{\beta} \Rightarrow t_{\hat{\alpha}} \) and \( t_{\hat{\beta}} \) (possibly noncentral) student t distributions with \( n - 2 \) and \( n - 3 \) dof.

3. **Approximate** by \( N(\mu, 1) \) and \( N(0, 1) \)
   \( \hat{\alpha} \) and \( \hat{\beta} \) are uncorrelated (but \( \hat{\beta} \) is mixed Gaussian)

4. **Determine a critical region** with correct coverage for any \( \alpha \) and \( \beta \) by defining boundary function \( g(\cdot) \)
Critical Region: New Approach
Principal elements

1. Symmetry
   (problem is symmetric for $\alpha = 0$ and $\beta$ varying, or $\beta = 0$ and $\alpha$ varying)
Critical Region: New Approach
Principal elements

1. Symmetry
   (problem is symmetric for $\alpha = 0$ and $\beta$ varying, or $\beta = 0$ and $\alpha$ varying)

2. Similarity: null rejection probability should not depend on the true values of $\alpha$ or $\beta$
Critical Region: New Approach
Principal elements

1. Symmetry
   (problem is symmetric for $\alpha = 0$ and $\beta$ varying, or $\beta = 0$ and $\alpha$ varying)

2. Similarity: null rejection probability should not depend on the true values of $\alpha$ or $\beta$

3. Optimize power in $45^0 \ (\text{mod } 90^0)$ directions because $\alpha$ and $\beta$ are an equal distance away from the $\alpha$- and $\beta$-axes
Critical Region: New Approach

Principal elements

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   (problem is symmetric for $\alpha = 0$ and $\beta$ varying, or $\beta = 0$ and $\alpha$ varying)

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- Solution $g(\cdot)$
Solution
Solution: Null Rejection Probabilities

\[ P[\text{Rej}] : \text{Size} \ (0^\circ \ \text{direction}) \]

\[
\begin{array}{cccc}
\beta/\sigma & 0 & 0.02 & 0.04 \\
P[\text{Rej}] : \text{Size} & 0.06 & 0.06 & 0.06 \\
\end{array}
\]
Solution: NRP

Size g 32 pts

\begin{align*}
\beta / \sigma &= 0.0485 \\
&= 0.0490 \\
&= 0.0495 \\
&= 0.0500 \\
&= 0.0505 \\
&= 0.0510 \\
&= 0.0515 \\
&= 0.0520
\end{align*}
Power Surface Solution
Superiority Solution

P[Rej]: Power in 45° direction

β/σ

0.2
0.4
0.6
0.8
1.0

P[Rej]: Power in 45° direction

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Two Important Issues

1. Non-existence Similar test $\Leftarrow ? \Rightarrow$ closeness of solution $g()$
Two Important Issues

1. Non-existence Similar test $\iff \Rightarrow$ closeness of solution $g()$
2. Power Envelope
Two Important Issues

1. Non-existence Similar test $\iff ? \Rightarrow$ closeness of solution $g()$

2. Power Envelope

1. There a sequence of functions $g_q(t)$ s.t. $\max |NRP(\beta) - 0.05| < \epsilon$
   What is the limit of $\epsilon$?
Two Important Issues

1. Non-existence Similar test $\leftrightarrow \Rightarrow$ closeness of solution $g()$

2. Power Envelope

1. There a sequence of functions $g_q(t)$ s.t. $\max |NRP(\beta) - 0.05| < \varepsilon$
   What is the limit of $\varepsilon$?

2. New test better than Wald, LR and LM
   Close to optimal?
Symmetry of the problem: if \((t_1, t_2)\) in CR \(\Rightarrow\) 8 points in CR
Sample points \(\{(±t_1, ±t_2), (±t_2, ±t_1)\}\) are all equivalent (same LR value)
Symmetry of the problem: if \((t_1, t_2)\) in CR \(\Rightarrow\) 8 points in CR
Sample points \(\{(\pm t_1, \pm t_2), (\pm t_2, \pm t_1)\}\) are all equivalent (same LR value)

Relevant density for symmetric Neyman-Pearson CR
\[
f(t_1, t_2; \mu_1, \mu_2) = \frac{1}{8} \cdot \\
\phi(t_1 - \mu_1) \phi(t_2 - \mu_2) + \phi(-t_1 - \mu_1) \phi(t_2 - \mu_2) \\
+ \phi(t_1 - \mu_1) \phi(-t_2 - \mu_2) + \phi(-t_1 - \mu_1) \phi(-t_2 - \mu_2) \\
+ \phi(t_2 - \mu_1) \phi(t_1 - \mu_2) + \phi(-t_2 - \mu_1) \phi(t_1 - \mu_2) \\
+ \phi(t_2 - \mu_1) \phi(-t_1 - \mu_2) + \phi(-t_2 - \mu_1) \phi(-t_1 - \mu_2)
\]

Note that: \(f(t_1, t_2; \mu_1, \mu_2)\):
Symmetry of the problem: if \((t_1, t_2)\) in CR \(\Rightarrow\) 8 points in CR
Sample points \{\((\pm t_1, \pm t_2), (\pm t_2, \pm t_1)\)\} are all equivalent (same LR value)

Relevant density for symmetric Neyman-Pearson CR
\[
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\phi(t_1 - \mu_1) \phi(t_2 - \mu_2) + \phi(-t_1 - \mu_1) \phi(t_2 - \mu_2) + \phi(t_1 - \mu_1) \phi(-t_2 - \mu_2) + \phi(-t_1 - \mu_1) \phi(-t_2 - \mu_2) + \phi(t_2 - \mu_1) \phi(t_1 - \mu_2) + \phi(-t_2 - \mu_1) \phi(t_1 - \mu_2) + \phi(t_2 - \mu_1) \phi(-t_1 - \mu_2) + \phi(-t_2 - \mu_1) \phi(-t_1 - \mu_2)
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Note that: \(f(t_1, t_2; \mu_1, \mu_2)\):

- is a proper density that integrates to 1
Symmetry of the problem: if \((t_1, t_2)\) in CR \(\Rightarrow\) 8 points in CR
Sample points \{\((\pm t_1, \pm t_2), (\pm t_2, \pm t_1)\)\} are all equivalent (same LR value)

Relevant density for symmetric Neyman-Pearson CR
\[
f(t_1, t_2; \mu_1, \mu_2) = \frac{1}{8} \cdot
\]
\[
\phi(t_1 - \mu_1) \phi(t_2 - \mu_2) + \phi(-t_1 - \mu_1) \phi(t_2 - \mu_2)
+ \phi(t_1 - \mu_1) \phi(-t_2 - \mu_2) + \phi(-t_1 - \mu_1) \phi(-t_2 - \mu_2)
+ \phi(t_2 - \mu_1) \phi(t_1 - \mu_2) + \phi(-t_2 - \mu_1) \phi(t_1 - \mu_2)
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basis for Point Optimal Invariant test
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Rationale for Point Optimal Invariant test
\( f(t_1, t_2; \mu_1, \mu_2) \) has a
• $f(t_1, t_2; \mu_1, \mu_2)$ has a
• Single mode for $\mu$ in a neighbourhood of zero $\mu_1, \mu_2 \approx 0$,
- $f(t_1, t_2; \mu_1, \mu_2)$ has a
- Single mode for $\mu$ in a neighbourhood of zero $\mu_1, \mu_2 \approx 0$,
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• Eight symmetric modes if $0 << \mu_2 << \mu_1$
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- Four (symmetric) modes if $\mu_1 = \mu_2 >> 0$.
- Eight symmetric modes if $0 << \mu_2 << \mu_1$
- The relevant LR (without loss of generality we can consider $\mu_2 = 0$ under $H_0$)
POIS: reject for large values of likelihood ratio:

\[
2 \log \left[ \frac{f(t_1, t_2; \mu_1, \mu_2)}{f(t_1, t_2; \mu_0, 0)} \right] = 2 \log \frac{e^{\mu_0(t_1+t_2)} e^{\frac{1}{2}(-\mu_1^2 - \mu_2^2 + \mu_0^2)}}{(e^{\mu_0 t_1} + e^{\mu_0 t_2}) (e^{\mu_0(t_1+t_2)} + 1)} \cdot (\cosh(\mu_2 t_1) \cosh(\mu_1 t_2) + \cosh(\mu_1 t_1) \cosh(\mu_2 t_2))
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POIS: reject for large values of likelihood ratio:

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The symmetries in $LR(t_1, t_2, \mu_0, \mu_1, \mu_2)$ follow from the symmetries in $f(t_1, t_2, \mu_1, \mu_2)$.
POIS: reject for large values of likelihood ratio:

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Parameter points \( \{(\pm \mu_0, \pm \mu_1, \pm \mu_2), (\pm \mu_0, \pm \mu_2, \pm \mu_1)\} \), are equivalent.
Critical Region POIS (1,1,1)
Critical Region POIS (1,2,2)
Point Optimal Invariant Similar test

Critical Region POIS (0.71, 0.74, 0.74)
Critical Region POIS $(1.0, 0.5, 0.5)$
• If $\mu_0 < \mu_1 = \mu_2$ then the LR test will reject for large values of $t_1$ and $t_2$. 

If $\mu_0 > \mu_1 = \mu_2$ then the LR test will reject values of $t_1$ and $t_2$ close to, and including the origin. This is undesirable. Consider only $\mu_0 < \mu_1 \geq \mu_2$. 

How to construct weighted null and alternative densities? Our solution is based on likelihood ratio.
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How to construct weighted null and alternative densities?

Our solution is based on likelihood ratio
$H_0 : \theta_1 \theta_2 \theta_3 = 0$

$H_0 : \theta_1 = 0 \vee \theta_2 = 0 \vee \theta_3 = 0$
\begin{align*}
    H_0 & : \theta_1 \theta_2 \theta_3 = 0 \\
    H_0 & : \theta_1 = 0 \lor \theta_2 = 0 \lor \theta_3 = 0 \\
    t & = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mu, I_3)
\end{align*}
\[ H_0 : \theta_1 \theta_2 \theta_3 = 0 \]
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For \( t_1 \rightarrow \infty \): Reduce to 2-dimensional solution \( g(\cdot) \)
Solution 3D

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\]

- For \( t_1 \to \infty \): Reduce to 2-dimensional solution \( g(\cdot) \)
- For \( t_1, t_2 \to \infty \): Reduce to 1-dimensional solution \( |t_3| > 1.96 \)
Solution: size
Conclusion

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Empirically Important problem of Mediation testing

- Poor power and size properties near the origin
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   - Poor power and size properties near the origin
   - Heavily dependent on (unknown) parameter values
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2 Theoretically Interesting

3 Practical Solution
   in terms of Critical Region

4 Uniformly 5% size (within $10^{-4}$)

5 Uniformly more powerful than all other 5% tests in the literature
Conclusion

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