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**A theoretical foundation for the Nelson and Siegel class of
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Abstract

This article establishes that most yield curve models within the popular Nelson and Siegel (1987, hereafter NS) class may be obtained as a formal Taylor approximation to the dynamic component of the generic Gaussian affine term structure model outlined in Dai and Singleton (2002). That fundamental theoretical foundation provides an assurance to users of NS models that they correspond to a well-accepted set of principles and assumptions for modeling the yield curve and its dynamics. Indeed, arbitrage-free NS models will parsimoniously and reliably represent the data generated by any Gaussian affine term structure model regardless of its true number of underlying factors and specification, and even non-arbitrage-free NS models will adequately capture the dynamics of the state variables. Combined with the well-established practical benefits of applying NS models, the theoretical foundation provides a compelling case for applying NS models as standard tools for yield curve modeling and analysis in economics and finance. As an illustration, this article develops a two-factor arbitrage-free NS model and applies it to testing for changes in United States yield curve dynamics. The results confirm those of Rudebusch and Wu (2007) based on a latent two-factor essentially affine term structure model: there was a material change in the behavior of the yield curve between the sample prior to 1988 and the sample from 1988 onwards.

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1 Introduction

This article establishes a theoretical foundation for the popular Nelson and Siegel (1987, hereafter NS) class of yield curve models via the generic Gaussian affine term structure model (hereafter GATSM) outlined in Dai and Singleton (2002). In particular, the article explicitly shows how the Level, Slope, and Curvature components that are common to all models of the NS class arise as “optimal approximations”, in a sense defined further below, to the dynamic component of the generic GATSM.

The primary motivation for establishing this result is to tackle “the elephant in the room” issue that has been conveniently overlooked with the extensive application of NS models since their inception.¹ That is, the Level, Slope, and Curvature components common to all models of the NS class were only justified heuristically when first proposed. That basis is highlighted with a selection of quotations from the introduction and conclusion of the original NS article:

“The purpose of this paper is to introduce a simple, parsimonious model that is flexible enough to represent the range of shapes generally associated with yield curves: monotonic, humped, and S shaped.” “A class of functions that readily generates the typical yield curve shapes is that associated with solutions to differential or difference equations. The expectations theory of the term structure provides heuristic motivation for investigating this class since, if spot rates are generated by a differential equation, then forward rates, being forecasts, will be the solution to the equations.”² “A more parsimonious model that can generate the same range of shapes is given by the solution equation for the case of equal roots.” “Our objective in this paper has been to propose a class of models, motivated by but not dependent on the expectations theory of the term structure, that offers a parsimonious representation of the shapes traditionally associated with yield curves.”

Even the recent introduction of arbitrage-free NS models (e.g. Sharef and Filipović 2004, Krippner 2006a, Christensen, Diebold and Rudebusch 2009, 2010), while at least imposing theoretical self-consistency by explicitly accounting for assumed Gaussian yield curve dynamics, still take the NS components as given.³ That is invariably justified by appealing to the practical benefits of NS models, such as their ease of estimation, close fit to the yield curve data, and intuitive estimated components.⁴ Of course, a rigorous theoretical foundation would be a far more compelling justification for NS models, and so the generic GATSM framework is used in this article to explicitly

¹Bank for International Settlements (2005) provides an overview of routine central bank use of NS models and Coroneo, Nyholm and Vidova-Koleva (2008) notes their widespread use by financial market practitioners. Diebold, Piazzesi and Rudebusch (2005) summarizes some recent time series applications and more examples of NS model applications are referenced later in the present article.

²The forward rate curve is then integrated to obtain the interest rate curve that is applied to observed yield curve data.

³For example, from Christensen, Diebold and Rudebusch 2010 footnote 4: “Our strategy is to find the affine AF model with factor loadings that match Nelson-Siegel exactly.”

⁴Dahlquist and Svensson (1996) originally advocated the suitability of NS models for monetary policy purposes on these grounds. Diebold and Li (2006) extends the discussion to the dynamic setting.

establish that the NS components and their dynamics correspond to the well-accepted set of principles and assumptions for modeling the yield curve and its dynamics.

The second motivation, which arises as a corollary to the theoretical foundation, is to establish a case for applying NS models as standard tools for yield curve modeling and analysis in economics and finance, rather than latent-factor GATSMs often applied in practice. One perspective for this case is that estimating latent-factor GATSMs is often a relatively complex process; for example Rudebusch and Wu (2007) p. 406 mentions “it should be noted that estimation of the standard no-arbitrage latent factor model, which is highly nonlinear, often appears to be plagued by numerical problems.” Another perspective is that the assumptions and restrictions often associated with estimating latent-factor GATSMs may lead to model mis-specifications and questionable economic interpretations. For example, Kim and Orphanides (2005) pp. 10-11 notes that the “rather arbitrary procedure” of setting statistically-insignificant parameters to zero (commonly used to reduce apparent overparametrization) risks “introducing significant biases in the resulting estimated model”, and can potentially lead to “different conclusions with the same [original] specification”. Conversely, the theoretical foundation for NS models gives an assurance that the underlying data-generating processes (hereafter DGPs) will be represented reliably, even if they contain many underlying factors with complex interactions, and that gives the NS components and parameters a clear economic interpretation.

The overview of how the theoretical foundation is obtained in sections 2 to 4 is as follows. Section 2 specifies the generic GATSM from Dai and Singleton (2002) and then derives the associated forward rate curve. Section 3 shows how the original NS forward rate curve arises from the dynamic component of the generic GATSM forward rate curve using an “optimal approximation”; specifically a low-order Taylor expansion around central measures of the eigenvalues associated with the generic GATSM. The NS Level component and its associated coefficient are therefore shown to correspond to the persistent (i.e. slowly mean-reverting, or near-zero eigenvalue) components of the generic GATSM, and the NS Slope and Curvature components are shown to correspond to the non-persistent (i.e. non-zero eigenvalue) components of the generic GATSM. In light of this example, section 4 discusses how most models within the NS class, with one notable exception being the Svensson (1995)/NS model, can be classified as varying optimal approximations to the generic GATSM. That classification system provides useful guidance to selecting the appropriate NS model for the given application, and for interpreting its output. In particular, it is theoretically valid in many cases to apply standard NS models rather than their more complex arbitrage-free counterparts.

The overview of case for applying NS models rather than latent-factor GATSMs as standard tools for yield curve modeling is given in sections 5 and 6. Section 5 discusses the case for the NS class of models in general, highlighting that while both provide approximations to the assumed “true” Gaussian DGP, NS models have distinct advantages from the application, transparency, and economic interpretation perspectives already mentioned above. Section 6 reinforces the case for those relative advantages in detail by developing a two-factor arbitrage-free NS model and applying it to investigate changes in United States yield dynamics. That topic was previously explored by Rudebusch and Wu (2007) with a latent two-factor essentially affine term structure model, and so enables an explicit comparison to the NS model and its application. Section 7 concludes.

2 The generic Gaussian affine term structure model

The generic GATSM specified in this section parallels the standard multifactor Gaussian dynamic term structure model as outlined in appendix A of Dai and Singleton (2002). It is the fully Gaussian subset of the affine framework outlined in Duffie and Kan (1996) with the essentially affine specification of market prices of risk from Duffie (2002).⁵

Three points of context for this article are worth noting up front. First, while the state variables are completely generic, and so could represent points on the yield curve as in Duffie and Kan (1996), it is more convenient for the subsequent discussion in section 5 to consider them as (potentially unobserved) economic and financial factors within the underlying economy. This follows the Duffie and Kan (1996) p. 321 interpretation that the state variables in an affine model can always, under standard assumptions, be related back to economic factors (e.g. preferences, technology, consumption, inflation, etc.) within a general equilibrium model. For example, Berardi and Esposito (1999) provides a generic basis for multifactor GATSMs based on an economy of the Cox, Ingersoll and Ross (1985*a*) type. Regarding financial factors (e.g. default risk, liquidity risk, repurchase effects, etc.), Duffie and Singleton (1999) shows how they can be incorporated into the generic GATSM framework,⁶ and Singleton (2006) chapter 14 contains an extensive summary of that literature.

Second, to make the exposition more transparent from the perspective of the original NS model, this article derives and works with the forward rate curve associated with the generic GATSM. The affine term structure literature more commonly uses bond prices and/or interest rate curves, but all are within an elementary transformation of each other and are equivalent perspectives for representing the yield curve.

Third, being fully Gaussian, the results for relating the generic GATSM to NS models do not extend to term structure models with full Cox, Ingersoll and Ross (1985*b*)/square-root dynamics. Appendix A illustrates this by example, and briefly discusses the practical implications. In short, NS models inherit the same theoretical shortcomings of GATSMs (i.e. positive probabilities of negative interest rates and constant volatilities), and that perspective should be considered when deciding if it is appropriate to apply an NS model for the task at hand. That said, the assumption of Gaussian dynamics is standard in economics and macrofinance (whether explicitly, or implicitly via the application of Gaussian-based econometrics),⁷ and this article presupposes from this point onward that the user has already made that assumption for the yield curve DGP.

Define the instantaneous short rate at time t as $r(t) = \xi_0 + \xi_1'X(t)$, where ξ_0 is a constant, $X(t)$ is an $N \times 1$ vector of state variables, and ξ_1 is a constant $N \times 1$ vector. Under the physical P measure, the state variables follow the process $dX(t) = K_P[\theta_P - X(t)]dt + \Sigma dW_P(t)$, where K_P is a constant $N \times N$ mean-reversion matrix, θ_P is a constant steady-state $N \times 1$ vector for $X(t)$, Σ is a constant $N \times N$ volatility matrix, and $W_P(t)$ is an $N \times 1$ vector of independent Brownian motions. Define the market prices of risk as $\Pi(t) = \Sigma^{-1}[\pi_0 + \pi_1X(t)]$, where π_0 is a constant $N \times 1$ vector and π_1 is

⁵In the notation of Dai and Singleton (2000) the specification is $A_0(N)$.

⁶While Duffie and Singleton (1999) focuses on default risk, pp. 193-94 of that article notes that other financial factors may also be treated in a similar manner.

⁷For example, all of the macrofinance models summarized in Rudebusch (2010) are specified with Gaussian innovations.

a constant $N \times N$ matrix. Under the risk-neutral Q measure, the state variables follow the process $dX(t) = K_Q [\theta_Q - X(t)] dt + \Sigma dW_Q(t)$, where $dW_Q(t) = dW_P(t) + \Pi(t) dt$, $K_Q = K_P + \pi_1$, and $\theta_Q = (K_P + \pi_1)^{-1} (K_P \theta_P - \Sigma \pi_0)$. Zero-coupon bond prices under measure Q for the generic GATSM are $P(t, T) = \exp [A^*(\tau) + B(\tau)' X(t)]$, where T is the time of maturity, $B(\tau) = [\exp(-K_Q' \tau) - I] (K_Q')^{-1} \xi_1$, τ is the time to maturity $\tau = T - t$ ($T \geq t$, $\tau \geq 0$), and I is the $N \times N$ identity matrix. The full expression for $A^*(\tau)$ is provided in Dai and Singleton (2002), but the present article requires only the summary results that $A^*(\tau)$ is required for the system to be arbitrage free, and it can be expressed in the functional form $-\xi_0 \tau + A(\tau)$.

From Heath, Jarrow and Morton (1992, hereafter HJM), instantaneous forward rates are defined as $f(t, T) = -\partial \log P(t, T) / \partial T$. Therefore, under measure Q , the generic GATSM forward rate curve is:

$$f(t, T) = \xi_0 + [\exp(-K_Q' \tau) \xi_1]' X(t) - \frac{\partial}{\partial \tau} A(\tau) \quad (1)$$

Now express K_Q' in eigensystem form; i.e. $K_Q' = Z \Psi Z^{-1}$, where Z is the $N \times N$ non-singular matrix of eigenvectors each normalized to 1, and Ψ is the $N \times N$ diagonal matrix containing the N eigenvalues $(\lambda_1, \dots, \lambda_n, \dots, \lambda_N)$. The latter are assumed to be unique and positive, which follows the standard assumption in Duffie and Kan (1996) and Dai and Singleton (2002). Hence, $\exp(-K_Q' \tau) = \exp(-Z \Psi Z^{-1} \tau) = Z \exp(-\Psi \tau) Z^{-1} = Z \Lambda Z^{-1}$, where $\Lambda = \text{diag}[\exp(\lambda_1 \tau), \dots, \exp(\lambda_n \tau), \dots, \exp(\lambda_N \tau)]$, an $N \times N$ diagonal matrix. The forward rates in equation 1 are then $f(t, T) = \xi_0 + \{Z \Lambda Z^{-1} \xi_1\}' X(t) - \frac{\partial}{\partial \tau} A(\tau)$. This can be expressed equivalently as:

$$f(t, T) = \xi_0 + \sum_{n=1}^{n_0} q_n(t) \exp(-\lambda_n \tau) + \sum_{n=n_0+1}^N q_n(t) \exp(-\lambda_n \tau) - \frac{\partial}{\partial \tau} A(\tau) \quad (2)$$

where the coefficients $q_n(t)$ associated with each unique $\exp(-\lambda_n \tau)$ represent the collection of coefficients of the $\exp(-\lambda_n \tau)$ terms that arise from the full matrix multiplication of $\{Z \Lambda Z^{-1} \xi_1\}' X(t)$.

For use in the example of the following section (but without loss of generality) it is assumed that the $q_n(t) \exp(-\lambda_n \tau)$ components have been re-ordered from the smallest to the largest eigenvalue, and then divided into two groups. The first group contains the components with eigenvalues λ_1 to λ_{n_0} that are close to zero (i.e. the persistent components, given they will have a slow exponential decay by time to maturity τ) and the second group contains the eigenvalues λ_{n_0+1} to λ_N that are not close to zero (i.e. non-persistent components).

Also for use in the following section, the first three components of equation 2 are collectively denoted the dynamic component of the generic GATSM. This terminology reflects that the time series properties of the generic GATSM forward rate curve are completely contained in the $q_n(t) \exp(-\lambda_n \tau)$ components. That is, the coefficients $q_n(t)$ are linear combinations of the original state variables $X(t)$, and the functions $\exp(-\lambda_n \tau)$ determine the expected (as at time t) evolution of forward rates from time t to time $t + \tau$. As time evolves, the stochastic term $\Sigma dW_P(t)$ for the generic GATSM imparts innovations to the state variables $X(t)$, which are reflected as innovations to the coefficients $q_n(t)$. Conversely, the non-dynamic component $\frac{\partial}{\partial \tau} A(\tau)$ is a time-

invariant function of time to maturity.

3 The generic GATSM to the original NS model

A formal Taylor approximation may be used to reproduce the original NS model from the exact expression of the dynamic component of the generic GATSM forward rate curve in equation 2. The treatment of the time-invariant component $\frac{\partial}{\partial \tau} A(\tau)$ is discussed further below.

For the first group of eigenvalues where $\lambda_n \simeq 0$, the first term of the Taylor expansion is $\exp(-\lambda_n \tau) \simeq 1$. For the second group of eigenvalues where $\lambda_n \gg 0$, express them relative to $\phi = \text{mean}(\lambda_{n_0+1}, \dots, \lambda_N)$,⁸ so that $\lambda_n = \phi(1 - \delta_n)$ and $\exp(-\lambda_n \tau) = \exp(-\phi \tau) \exp(\delta_n \phi \tau)$. To the latter expression, applying a first-order Taylor approximation around δ_n gives $\exp(-\phi \tau)(1 + \delta_n \phi \tau)$, so $q_n(t) \exp(-\lambda_n \tau) \simeq q_n(t) \exp(-\phi \tau) + q_n(t) \delta_n \phi \tau \exp(-\phi \tau)$. Substituting these results into equation 2 gives:

$$f(t, T) \simeq \xi_0 + \sum_{n=1}^{n_0} q_n(t) + \left[\sum_{n=n_0+1}^N q_n(t) \right] \exp(-\phi \tau) + \left[\sum_{n=n_0+1}^N q_n(t) \delta_n \right] \phi \tau \exp(-\phi \tau) \quad (3)$$

This expression is precisely the functional form of the original NS model of the forward rate curve, i.e.:

$$f(t, T) \simeq f_{\text{NS}}(t, \tau) = L(t) + S(t) \exp(-\phi \tau) + C(t) \phi \tau \exp(-\phi \tau) \quad (4)$$

where $f_{\text{NS}}(t, \tau)$ adopts the typical time and time to maturity notation for NS models, 1, $\exp(-\phi \tau)$, and $\phi \tau \exp(-\phi \tau)$ are the forward rate factor loadings for the original NS model, and $L(t) = \xi_0 + \sum_{n=1}^{n_0} q_n(t)$, $S(t) = \sum_{n=n_0+1}^N q_n(t)$, and $C(t) = \sum_{n=n_0+1}^N q_n(t) \delta_n$ are the coefficients for the original NS model. The standard transformation $R_{\text{NS}}(t, \tau) = \frac{1}{\tau} \int_0^\tau f_{\text{NS}}(t, \tau) d\tau$ then produces the original NS model for the interest rate curve.⁹

The exposition above shows that the original NS model parsimoniously approximates the constant plus the persistent dynamic components of the generic GATSM to zeroth order using the NS Level component, and approximates the non-persistent dynamic components to first order using the combination of NS Slope and Curvature components. Moreover, it is an “optimal approximation” in the sense that each additional NS component corresponds precisely to each successive term of the Taylor expansion of the dynamic GATSM components. The original NS model is therefore guaranteed to reliably represent all the non-persistent dynamic components of the generic GATSM to

⁸Any other central measure of $(\lambda_{n_0+1}, \dots, \lambda_N)$ would suffice for the exposition in this article. In practice, ϕ is an estimated parameter.

⁹That is, $R_{\text{NS}}(t, T) = L(t) + S(t) \left(\frac{1 - \exp(-\phi \tau)}{\phi \tau} \right) + C(t) \left(\frac{1 - \exp(-\phi \tau)}{\phi \tau} - \exp(-\phi \tau) \right)$. Interestingly, the original NS article, p. 475, also notes that “This model may also be derived as an approximation to the solution in the unequal roots case by expanding in a power series in the difference between the roots.” The connection to the present article is coincidental however; from a mathematical perspective, the second-order differential equation assumed in NS happens to produce the same general solution as a bivariate first-order differential equation, which would be a minimal multifactor GATSM without an allowance for stochastic dynamics. From an economic perspective, it would be theoretically untenable to propose an N^{th} -order differential equation as the basis for a generic interest rate model.

first order and only omit second-order and higher terms. It is worth stressing the sense in which NS components are the optimal approximation by highlighting that any other set of approximating functions (e.g. see James and Webber (2000) chapter 15) cannot precisely represent the Taylor expansion terms around the central eigenvalues for the dynamic component of the generic GATSM term structure. For example, a third-order polynomial function would not provide a precise third-order approximation (or even a precise zeroth-order approximation) of the non-persistent GATSM components.

The original NS model as derived above can be made arbitrage free (if required; see the discussion at the end of the following section) by adding appropriate terms to account for the effects that the market prices of risk and volatilities of the NS factor loadings have on the forward rate curve. For example, Christensen, Diebold and Rudebusch (2010) directly derives the arbitrage-free (hereafter AF) terms for $R_{NS}(t, T)$ via a particular three-factor GATSM designed to reproduce the original NS factor loadings. That AF/NS model has up to 18 more parameters than just the single parameter ϕ in the original NS model.¹⁰

It is important to note that the calculation of the appropriate AF terms from a base NS model, by whatever means, is deliberate. This guarantees the resulting model will be AF with respect to the NS components that are themselves an optimal approximation of the dynamic component of the GATSM. Conversely, a direct Taylor approximation of the generic GATSM including its AF terms would not necessarily guarantee a resulting AF model.

4 A GATSM perspective for classifying, selecting, and applying NS models

By following the example in the previous section, it is possible to classify most NS models as a particular Taylor approximation of the generic GATSM. The key aspects are: (1) the number of groups of non-zero eigenvalues assumed for the non-persistent dynamic components of the generic GATSM, which determines how many mean eigenvalue parameters are required; (2) the order of approximation chosen around each mean eigenvalue, which determines the number of components from the Taylor expansion associated with each mean eigenvalue; and (3) whether the AF term is included, which determines if the NS model is AF with respect to its factor loadings. For example, the original NS model implicitly assumes the following: (1) one non-zero eigenvalue group, which gives the single mean eigenvalue parameter ϕ ; (2) a first-order Taylor approximation for the single non-zero eigenvalue group, which gives two terms (the Slope and Curvature components); and (3) no inclusion of the AF term, so the model is not AF (but it can be made so with the modification from Christensen et al. 2010).

The various permutations of each of the three aspects above can obviously generate a wide variety of NS models, but this section discusses just the range of NS models already in use and then three parsimonious variants that naturally suggest themselves.

The Christensen, Diebold and Rudebusch (2009)/NS model is the most comprehensive model within the NS class to date. It has the forward rate form $f(t, \tau) = L(t) +$

¹⁰Krippner (2006a) is an analogous AF version of the original NS model, but assumes independent state variable innovations and constant market prices of risk.

$S_1(t) \exp(-\phi_1\tau) + C_1(t) \phi_1\tau \exp(-\phi_1\tau) + S_2(t) \exp(-\phi_2\tau) + C_2(t) \phi_2\tau \exp(-\phi_2\tau) + AF(\tau)$, where $AF(\tau)$ abbreviates the AF term from Christensen et al. (2009) in its forward rate form. The Christensen et al. (2009)/NS model therefore represents the persistent generic GATSM components to a zeroth-order approximation, the non-persistent components with two groupings of non-zero eigenvalues (i.e. $\phi_1 = \text{mean}(\lambda_{n_0+1}, \dots, \lambda_{n_1}) < \phi_2 = \text{mean}(\lambda_{n_1+1}, \dots, \lambda_N)$) each to a first-order approximation, and includes the appropriate AF terms to ensure the model is AF with respect to the five factor loadings. The Svensson (1995)/NS model omits the AF and $S_2(t) \exp(-\phi_2\tau)$ terms, and the Bliss (1997)/NS model further omits the $C_1(t) \phi_1\tau \exp(-\phi_1\tau)$ term. Sharef and Filipović (2004) directly derives a two-factor AF version of the Svensson (1995)/NS model, but notes (p. 698) that the associated process for the short rate is not Markovian.

At the other extreme, the Diebold, Li and Yue (2008)/NS model with the forward rate form $f(t) = L(t) + S(t) \exp(-\phi\tau)$ is the most parsimonious representation of the generic GATSM.¹¹ It represents both the persistent and non-persistent components of the generic GATSM to zeroth order and omits any AF adjustments. Interestingly, the two-factor HJM model (HJM pp. 91-92) is an AF version of the Diebold et al. (2008)/NS model assuming independent factor innovations, by coincidence of a constant and an exponential decay by time to maturity as the assumed HJM volatility functions for the forward rate curve. Section 6 effectively extends the HJM model to allow for correlation.

The Diebold et al. (2008)/NS model and its AF counterparts are therefore examples of “balanced” NS models, in the sense that the order of approximation is the same for the persistent and non-persistent components. Three other parsimonious balanced NS models that may be worth investigating in future work are: (1) approximate the persistent generic GATSM components to first order to match the first-order approximation inherent in the Slope and Curvature components,¹² resulting in $f(t, \tau) = L_1(t) + L_2(t) \tau + S(t) \exp(-\phi\tau) + C(t) \phi\tau \exp(-\phi\tau)$, where $L_1(t) = -\sum_{n=1}^{n_0} q_n(t) \lambda_n$; (2) approximate two groups of non-persistent components of the generic GATSM to zeroth order, resulting in $f(t, \tau) = L(t) + S_1(t) \exp(-\phi_1\tau) + S_2(t) \exp(-\phi_2\tau)$, where $0 < \phi_1 < \phi_2$; and (3) split the constant ξ_0 from the generic GATSM and allow the Level component to have a non-zero mean eigenvalue, resulting in $f(t, \tau) = \xi_0 + L(t) \exp(-\phi_1\tau) + S(t) \exp(-\phi\tau)$ where $0 \lesssim \phi_1 \ll \phi_2$ and $L(t) = \sum_{n=1}^{n_0} q_n(t)$. All of these models can be made AF with respect to their NS components as in Christensen et al. (2009, 2010), or directly via the HJM framework as in the example of section 6.

Choosing the particular NS model to apply has in the past been largely an empirical matter; i.e. trading off parsimony against goodness of fit to the yield curve data. However, the classification above suggests a systematic approach to introducing or omitting terms if maintaining a correspondence with the GATSM class is desired.

¹¹This assumes one component each to represent the near-zero and non-zero groups of eigenvalues from the generic GATSM, which is arguably the minimal model one would want for empirical work. An “over-parsimonious” NS model would be to use just a Slope component, in which case the AF version would be the Vasicek (1977) model. The absolutely most parsimonious NS model would be to use just a Level component. That is the basis for the traditional “duration” calculations from Macauley (1938) that are often used to gauge the price-sensitivity of interest rate securities to a level shift in the yield curve. The AF version would be the Vasicek (1977) model with the limit of a zero mean-reversion parameter.

¹²That is, $\exp(-\lambda_n\tau) \simeq 1 - \lambda_n\tau$, and so $\sum_{n=1}^{n_0} q_n(t) \exp(-\lambda_n\tau) \simeq \sum_{n=1}^{n_0} q_n(t) - \sum_{n=1}^{n_0} q_n(t) \lambda_n\tau$.

For example, the Svensson (1995)/NS model, its associated Sharf and Filipović (2004) AF extension, and the Bliss (1997)/NS models would be avoided because the second Curvature term $C_2(t)$ cannot by itself represent an unconstrained first-order approximation to the component of the generic GATSM associated with the second group of eigenvalues. Either the second Slope term $S_2(t)$ should also be added (creating the non-AF analogue of the Christensen, Diebold and Rudebusch (2009)/NS model), or the second Curvature term $C_2(t)$ dropped (recreating the original NS model). Another aspect is more subtle: from the strict perspective of maintaining a foundation within the generic GATSM, NS models should be applied with a constant parameter ϕ (or parameters ϕ_1, ϕ_2 , etc.) because that corresponds to the constant parameters assumed in the generic GATSM.¹³

Once the appropriate NS model has been chosen, consideration needs to be given to the associated AF term. The theoretical case for the latter was originally proposed in Björk and Christensen (1999) and further established in Filipović (1999, 2000). Ideally, the AF term should be included in empirical applications to maintain theoretical consistency between the cross-sectional and time-series properties of the given NS model, and the correspondence back to the GATSM. Explicit estimates of the market prices of risk and the volatilities associated with the AF property may also provide useful information to the user, and are certainly essential when pricing instruments that are heavily influenced by interest rate volatility, such as options on fixed interest securities.

However, standard NS models will provide output that is theoretically valid to use purely in a time series context, such as for forecasting the yield curve, establishing relationships with macroeconomic time-series data, or generating zero-coupon interest rate data to be used subsequently in a time series context. This result arises in principle because the AF volatility terms are time-invariant functions of time to maturity τ , and so cannot influence the time variability of the estimated coefficients for an NS model relative to its AF counterpart. The NS model coefficients will therefore be within a constant of the counterpart AF/NS model coefficients if the market prices of risk may be assumed constant, and the NS model coefficients will subsume the time-varying component of term premia from the counterpart AF/NS model specified with essentially affine market prices of risk.¹⁴

5 NS models versus latent-factor GATSMs

The corollary to the exposition so far is a compelling case for applying NS models as standard tools for yield curve modeling and analysis in economics and finance, rather than latent-factor GATSMs (hereafter LF/GATSMs). This section outlines the case in general for the NS class of models, while section 6 provides a detailed example

¹³While it would be tempting to interpret time variation in ϕ as representing time variation in the mean-reversion matrix K_Q , a generic GATSM that formally allowed for such flexibility would not necessarily result in factor loadings reducible to the NS form using the Taylor approximation approach as in section 3.

¹⁴A more formal exposition of this aspect and further discussion is omitted because it is not central to the present article. It is available from the author on request. Note also that, aside from purely theoretical considerations, the practical relevance of AF terms for NS models has been questioned in Coroneo et al. (2008).

of deriving and applying a two-factor AF/NS model that specifically illustrates the practical advantages relative to a two-LF/GATSM.

For the purpose of illustrating the general case, take an example where the user has already decided it appropriate to assume (as is common in practice) that the “true” DGP for a set of observed yield curve data has Gaussian dynamics. That assumption translates to an assumed N -factor GATSM, although N may be large given the many factors that can potentially influence the yield curve in practice. For example, Hördahl, Tristani and Vestin (2006) models the nominal government yield curve via the four economic factors of output growth, inflation, monetary policy, and an inflation target, but also notes (p. 408) that “The model is certainly too stylized - for example, in its ignoring foreign variables or the exchange rate - to provide a fully-satisfactory account of German macroeconomic dynamics.”. Additional financial market factors influencing the government yield curve could include liquidity and repurchase effects (see Fisher (2002) and Fleming (2003) respectively), and corporate yield curves can contain many more factors in addition to the government yield curve, including default risk, corporate bond liquidity, and the three Fama and French (1993) factors (see Elton, Gruber, Agrawal and Mann 2001). Furthermore, the N -factor GATSM may also contain complex interactions in its mean-reversion, market price of risk, and volatility specifications.

Now consider applying an NS model and an LF/GATSM with J factors to the data from the N -factor GATSM. J may be assumed to be much smaller than N , given that two- and three-LF/GATSMs are most commonly applied in practice. Indeed, the assumption of three factors is often justified on the grounds that three principal components explain the vast majority of U.S. government yield curve movements, but it is also effectively a practical limit for unrestricted or mildly-restricted LF/GATSMs. For example, Duffee (2002) notes on p. 418 that “Litterman and Sheinkman (1991) find that three factors explain the vast majority of Treasury bond price movements. This is fortunate, because general three-factor affine models are already computationally difficult to estimate owing to the number of parameters. Adding another factor would make this investigation impractical.”

Both the NS model and the J -LF/GATSM will provide approximations to the assumed N -factor GATSM, but the NS model will have distinct advantages from three perspectives. The first is the ease of estimation. NS models have the property of reliable convergence, and non-AF/NS model estimations often proceed via OLS or non-linear LS (depending on whether the yield curve data are zero-coupon yields or coupon-bearing bond prices). Conversely, estimating the J -LF/GATSM is likely to encounter the well-documented issues of multiple local maxima and overparametrization. A commonly employed resolution for the overparametrization issue is to impose prior restrictions on parameters (e.g. a diagonal innovation covariance matrix) and/or to use the “rather arbitrary procedure” of setting statistically-insignificant parameters to zero.¹⁵ However, as subsequently discussed below, such constraints have potential implications for the model output.

The second perspective is the transparency in the approximation. That is, the optimal approximation inherent in an NS model guarantees the N -factor GATSM will be represented to a known and precise order of approximation with just several com-

¹⁵Examples are Dai and Singleton (2002), Duffee (2002), and Rudebusch and Wu (2007). The quote is from Kim and Orphanides (2005) p. 10.

ponents, regardless of (or even without knowledge of) the actual number of state variables, their nature (e.g. inflation, default risk, etc.), and their interactions in the mean-reversion, market price of risk, and volatility specifications. Conversely, the assumptions often required to reduce overparametrization risk “introducing significant biases in the resulting estimated model” (Kim and Orphanides, 2005 p. 11), so it becomes unclear how exactly the J -LF/GATSM is representing the N -factor GATSM.

The third perspective is the economic interpretation of the output from the estimated models. Given the transparency of the approximation associated with any NS model, the user knows how the associated NS coefficients and their associated parameters correspond to the persistent and non-persistent components of the DGP underlying the yield curve data. The individual economic and financial factors obviously cannot be identified from the NS components, but relationships may nevertheless be suggested by the context of the application. For example, one might anticipate the NS Level coefficient for a nominal government yield curve to co-vary with inflation (a persistent macroeconomic variable), and the non-Level components to co-vary with output growth (a non-persistent macroeconomic variable). Indeed, such results have already been established empirically in Diebold, Rudebusch and Aruoba (2006) using the original NS model. Conversely, as discussed extensively in Collin-Dufresne, Goldstein and Jones (2008), the output from the J -LF/GATSM often does not have a clear economic interpretation, and multiple local maxima (with associated different parameter estimates) along with the sensitivity of the model to the assumptions often required to reduce overparametrization can potentially lead to “different conclusions with the same specification” (Kim and Orphanides, 2005 p. 11).

To complete this section, the following briefly discusses the performance of NS models versus LF/GATSM in cases where direct comparisons are available.¹⁶ In general, NS models have generally given similar or superior results to LF/GATSMs. Examples are: (1) fitting the yield curve (the original NS model has a similar fit to the three-LF/GATSM of Duffee, 2002);¹⁷ (2) forecasting the yield curve (e.g. Vincente and Tabak (2008) obtains lower forecast RMSEs with the original NS model than a three-LF/GATSM);¹⁸ (3) macrofinance (e.g. in the Diebold, Rudebusch and Aruoba (2006) application mentioned earlier, the parsimony of the original NS model allows for bidirectional dynamics between the yield curve and macroeconomic variables, while Diebold et al. (2005) notes that the complexity of the five-LF/GATSM from Ang and Piazzesi (2003) requires prior restrictions that limit the directionality from macroeconomic factors to yields); and (4) monitoring inflation compensation (e.g. using a modified AF/NS model to jointly model nominal and inflation-indexed yield curves, Christensen, Lopez and Rudebusch (2010) report better correlations between model-

¹⁶NS models have also applied successfully to other topics that have been investigated with LF/GATSMs, but the empirical results are not directly comparable due to sample differences. Examples include: (1) modeling non-government yields and liquidity conditions, e.g. Christensen, Lopez and Rudebusch (2009); (2) investigating the uncovered interest parity puzzle, e.g. Krippner (2006b) and Chen and Tsang (2009); (3) modeling value at risk in fixed-interest portfolios, e.g. Diebold, Ji and Li (2006); and (4) investigating international yield curve transmissions, e.g. Diebold et al. (2008).

¹⁷Fitting the original NS model to the Duffee (2002) US government yield curve data gave an RMSE of 7.3 basis points (bps), and an MAE of 4.8 bps. The respective quantities from Duffee (2002) are 10.8 bps and 5.3 bps. I thank Gregory Duffee for making the data available on his website.

¹⁸Diebold and Li (2006) also obtains lower forecast RMSEs than Duffee (2002), but the sample is different.

implied and surveyed 5-year inflation expectations than D’Amico, Kim and Wei (2008) which uses three-LF/GATSMs for both the nominal and inflation-indexed yield curves).

6 An application to U.S. yield curve dynamics

As an example to illustrate in detail the points made in the prior section, this section develops and applies a two-factor AF/NS model of the yield curve to investigate the dynamics of the U.S. yield curve from 1971 to 2010. Section 6.1 provides an overview of the yield curve data for that period. Section 6.2 derives and applies the two-factor AF/NS model assuming constant market prices of risk, and is denoted the AF/NS(2) model. Section 6.3 extends the model to an essentially affine specification for the market prices of risk, and is denoted the EA/AF/NS(2) model.

The behavior of U.S. yield curve dynamics has already been investigated in Rudebusch and Wu (2007, hereafter RW) using a two-LF/GATSM with essentially affine market prices of risk. That application, using data from 1971 to 2002, may be compared directly to the application of the EA/AF/NS(2) model over the sample periods A and B outlined below.

6.1 Yield curve data

Figure 1 provides an overview of the U.S. yield curve and its dynamics by plotting the 3-month and 15-year government-risk interest rate data (as detailed further below), and also the spread between those two rates. The latter is defined opposite to the common convention of a long-maturity rate less a short-maturity rate so it coincides qualitatively with the inverted shape of the Slope function in the following NS models. Hence, the troughs represent periods of easy monetary policy and values above zero represent an inverted yield curve.

The full sample period is divided into three samples. Sample A is from November 1971 (the beginning of the 15-year data) to December 1987 (to match the end of sample A from RW). Sample B is from January 1988 to December 2002 (to match sample B from RW). Sample C, from January 2003 to June 2010 (the latest data available at the time of the analysis), simply contains the additional data available relative to RW, although it is arguably a unique period in its own right because it conveniently begins amid the onset of U.S. deflation concerns in late-2002/early-2003 (e.g. see Billi, 2009 p. 83) and ends with the ultra-easy U.S. monetary policy following the 2007/2008 global financial crisis. That said, the results for sample C are reported only out of interest and for comparison to the estimates over samples A and B, and should not be taken as an advocacy to apply NS models over this period. It would be theoretically questionable to represent the nominal yield curve in a low to near-zero interest rate environment with a yield curve model that cannot respect the zero bound for nominal interest rates.

The maturity span of the available yield curve data changes over the full sample, reflecting the longest-maturity bond on issue at any point in time. Sample A uses 3- and 6-month Treasury bill rates (from the Federal Reserve Economic Database on the St Louis Federal Reserve website, converted to a continuously compounding basis) and the 1-, 2-, 3-, 4-, 5-, 7-, 10-, and 15-year continuously compounding zero-coupon

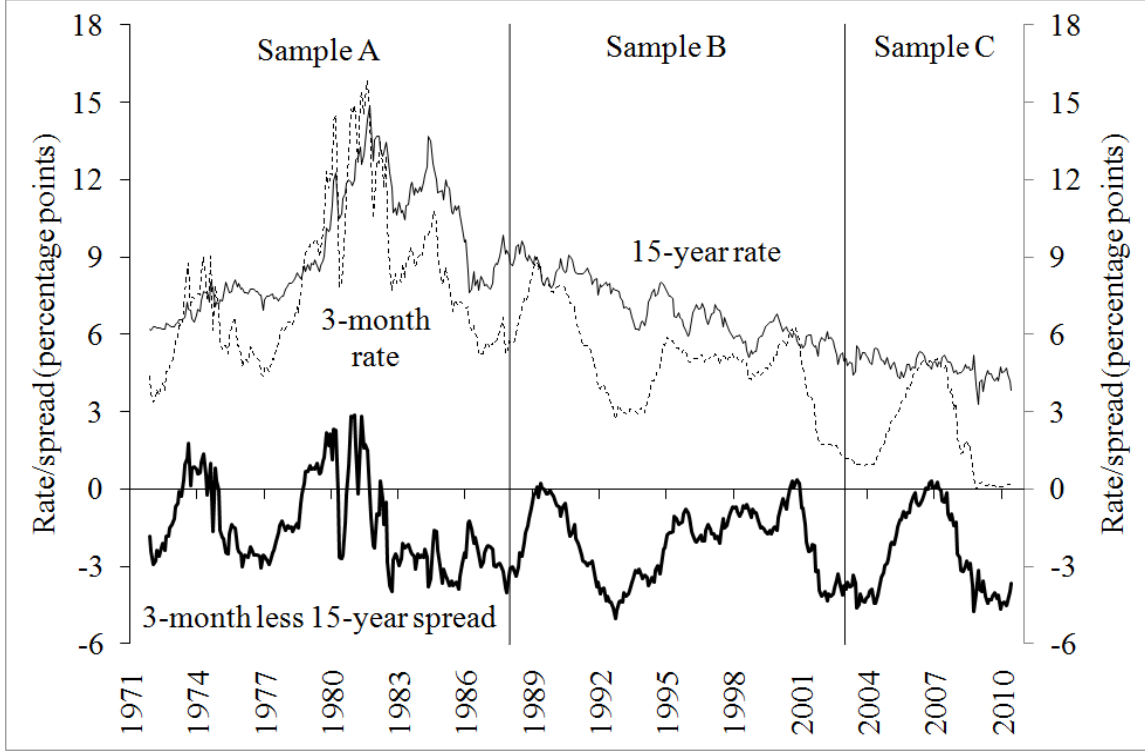


Figure 1: The 3-month and 15-year continuously compounding zero-coupon interest rate data and the spread between those two rates.

government interest rates from the data set described in Gürkaynak, Sack and Wright (2008). Sample B is estimated with data of the same maturity span (hereafter denoted 3-m/15-y) to allow a direct comparison to sample A, and also with the addition of the Gürkaynak et al. (2008) 20- and 30-year data (which became available in July 1981 and November 1985 respectively). Sample C is estimated with data of the latter maturity span (hereafter denoted 3-m/30-y) to allow a direct comparison to sample B. All of the data are month-end rates taken from the original sets of daily data.

6.2 The AF/NS(2) model

6.2.1 Deriving the AF/NS(2) model

The NS model with two factors is $\beta_1(t) + \beta_2(t) \cdot \exp(-\phi\tau)$. The HJM framework with the modification from Tchuindjo (2008) allows the factors to have innovations $\Omega [dW_1(t), \exp(-\phi\tau) \cdot dW_2(t)]$ with a non-zero correlation, i.e.

$$\Omega = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \quad (5)$$

where σ_1 and σ_2 are the factor volatilities (annualized standard deviations of the factor innovations), ρ is the correlation of factor innovations, and $dW_1(t)$ and $dW_2(t)$ are independent Wiener increments.

The expression for the AF/NS(2) forward rate curve in the Tchuindjo (2008)/HJM

framework is therefore:

$$\begin{aligned}
f(t, \tau) &= [1, \exp(-\phi\tau)] \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \end{bmatrix} \\
&+ \int_0^\tau [\sigma_1, \sigma_2 \exp(-\phi s)] \begin{bmatrix} \gamma_{0,1} \\ \gamma_{0,2} \end{bmatrix} ds \\
&- \int_0^\tau \left\{ [\sigma_1, \sigma_2 \exp(-\phi\tau)] \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \left(\int_s^\tau \begin{bmatrix} \sigma_1 \\ \sigma_2 \exp(-\phi u) \end{bmatrix} du \right) ds \right\} \quad (6)
\end{aligned}$$

where the first line is $\mathbb{E}_t[r(t+\tau)]$, the expected value, conditional upon information available at time t , of the instantaneous short rate at time $t+\tau$; the second line is the term premium function involving the factor volatilities and the constant market prices of risk $\gamma_{0,1}$ and $\gamma_{0,2}$; the third line is the volatility effect involving the factor volatilities and their correlation ρ ; and u and s are dummy integration variables. Note that $\gamma_{0,1}$ and $\gamma_{0,2}$ are defined in this article as positive quantities, so they (intuitively) add positive spreads to $\mathbb{E}_t[r(t+\tau)]$ and therefore the observed yield curve.

The solution to the forward rate expression in equation 6 is:¹⁹

$$\begin{aligned}
f(t, \tau) &= \beta_1(t) + \beta_2(t) \cdot \exp(-\phi\tau) \\
&+ \sigma_1 \gamma_{0,1} \cdot \tau + \sigma_2 \gamma_{0,2} \cdot F(\phi, \tau) \\
&- \sigma_1^2 \cdot \frac{1}{2} \tau^2 - \sigma_2^2 \cdot \frac{1}{2} [F(\phi, \tau)]^2 \\
&- \rho \sigma_1 \sigma_2 \cdot \tau F(\phi, \tau) \quad (7)
\end{aligned}$$

where $F(\phi, \tau) = \frac{1}{\phi} [1 - \exp(-\phi\tau)]$. The expression for $f(t, \tau)$ may be more conveniently expressed as $f(t, \tau) = a(\tau) + b(\tau) \beta(t)$, where $a(\tau) = \sigma_1 \gamma_{0,1} \cdot \tau + \sigma_2 \gamma_{0,2} \cdot F(\phi, \tau) - \sigma_1^2 \cdot \frac{1}{2} \tau^2 - \sigma_2^2 \cdot \frac{1}{2} [F(\phi, \tau)]^2 - \rho \sigma_1 \sigma_2 \cdot \tau F(\phi, \tau)$, $b(\tau) = [1, \exp(-\phi\tau)]$, and $\beta(t) = [\beta_1(t), \beta_2(t)]'$. The standard relationship $R(t, \tau) = \frac{1}{\tau} \int_0^\tau f(t, \tau) d\tau$ gives the interest rate function as $R(t, \tau) = \bar{a}(\tau) + \bar{b}(\tau) \beta(t)$, where $\bar{a}(\tau) = \sigma_1 \gamma_{0,1} \cdot \frac{1}{2} \tau + \sigma_2 \gamma_{0,2} \cdot \frac{1}{\phi} [1 - \frac{1}{\tau} F(\phi, \tau)] - \sigma_1^2 \cdot \frac{1}{6} \tau^2 - \sigma_2^2 \cdot \frac{1}{2\phi^2} (1 - \frac{1}{\tau} F(\phi, \tau) - \frac{1}{2\tau} \phi [F(\phi, \tau)]^2) - \rho \sigma_1 \sigma_2 \cdot \frac{1}{\phi^2} (1 - \frac{1}{\tau} F(\phi, \tau) + \frac{1}{2} \phi \tau - \phi F(\phi, \tau))$ and $\bar{b}(\tau) = [1, \frac{1}{\tau} F(\phi, \tau)]$.

An observation of zero-coupon continuously compounding yield curve data at time t may therefore be represented as:

$$R(t) = \bar{A} + \bar{B} \beta(t) + v(t) \quad (8)$$

where $R(t)$ is the $K \times 1$ vector of yield curve data (with $K = 10$ or 12 in the empirical application below), \bar{A} is the $K \times 1$ vector $[\bar{a}(\tau_1), \dots, \bar{a}(\tau_k), \dots, \bar{a}(\tau_K)]'$, and \bar{B} is the $K \times 2$ matrix $[\bar{b}(\tau_1), \dots, \bar{b}(\tau_k), \dots, \bar{b}(\tau_K)]$, where $\tau_1, \dots, \tau_k, \dots, \tau_K$ are the times to maturity of the yield curve data.

The evolution of $\beta(t)$ over a finite time step Δt may be derived directly as:

$$\beta(t + \Delta t) = \Phi(\phi, \Delta t) \beta(t) + \varepsilon(t + \Delta t) \quad (9)$$

where $\Phi(\phi, \Delta t) = \text{diag}[1, \exp(-\phi\Delta t)]$, a 2×2 diagonal matrix.

¹⁹ All the results in this and the following subsection follow from straightforward but tedious calculus and algebra. Full workings of all the results are contained in appendix B.

Equations 8 and 9 are a measurement and state equation that may be used in the Kalman filter to estimate the model. Regarding the additional elements required for the Kalman filter, the covariance matrix for the state equation may be evaluated as:

$$Q = \int_0^{\Delta t} \Phi(\phi, s) \Omega [\Phi(\phi, s)]' ds \quad (10)$$

$$= \begin{bmatrix} \sigma_1^2 \cdot \Delta t & \rho \sigma_1 \sigma_2 \cdot F(\phi, \Delta t) \\ \rho \sigma_1 \sigma_2 \cdot F(\phi, \Delta t) & \sigma_2^2 \cdot F(2\phi, \Delta t) \end{bmatrix} \quad (11)$$

and the covariance matrix for the measurement equation is assumed to take the form $H = \text{diag}[\sigma_v^2(\tau_1), \dots, \sigma_v^2(\tau_K)]$. That assumption is standard in the literature, as is the assumption that all other contemporaneous and intertemporal covariances are zero.

The starting values for the state variable vector and its covariance matrix are the unconditional values for the AF/NS(2) model, i.e. $\mathbb{E}[\beta(t)]$ and $\int_0^\infty \Phi(\phi, s) \Omega \Phi(\phi, s) ds$, except for the values purely associated with $\beta_1(t)$. The latter are undefined because $\beta_1(t)$ is a unit root process. Following the suggestion of Hamilton (1994) p. 378, an appropriate starting value with a variance reflecting its confidence interval may be substituted. Given that the Level component is reflecting the near unit root processes in the generic GATSM, the mean and variance for $\beta_1(t)$ are supplied as 0 and $\sigma_\eta^2 / (1 - \psi^2)$, where ψ is the coefficient from the autoregression $\beta_1^*(t + \Delta t) = \psi \cdot \beta_1^*(t) + \eta(t)$ and $\beta_1^*(t)$ is the Level coefficient series obtained from a preliminary estimation of the non-AF two-factor NS model by OLS.²⁰ Hence, the starting values are $\beta_{1|0} = [0, 0]'$ and:

$$P_{1|0} = \begin{bmatrix} \sigma_\eta^2 / (1 - \psi^2) & \rho \sigma_1 \sigma_2 \cdot \frac{1}{\phi} \\ \rho \sigma_1 \sigma_2 \cdot \frac{1}{\phi} & \sigma_2^2 \cdot \frac{1}{2\phi} \end{bmatrix} \quad (12)$$

Before proceeding to estimate the model, three observations detail how several estimation issues associated with LF/GATSMs have already been resolved with the AF/NS(2) model. The same observations also apply to the EA/AF/NS model in the following subsection.

First, the AF/NS(2) model and its Kalman filter set-up above is equivalent to the two-LF/GATSM and set-up from Babbs and Nowman (1999), but with a limit of zero mean reversion for one of the factors.²¹ That reduces by one the number of parameters to be estimated, but it more importantly imposes the restriction that the mean-reversion parameter for the second factor will always be greater than for the first. As mentioned in Collin-Dufresne et al. (2008) footnote 15, that restriction resolves the Babbs and Nowman (1999) identification problem that would otherwise exist for the two-LF/GATSM. The practical implication of the problem is that the two-LF/GATSM has two observationally equivalent maxima in its likelihood function

²⁰The parameter ψ was latter updated using the $\beta(t)$ coefficients obtained from an initial maximum likelihood estimation, and another maximum likelihood estimation was undertaken to provide the final estimates reported subsequently. However, the additional iterations made an immaterial difference to the initial results. The estimates of $\sigma_\eta^2 / (1 - \psi^2)$ were around 3 percentage points, which is quite diffuse (as would be expected from a near-unit-root process). A more formal approach would be to use the diffuse Kalman filter, as outlined, for example, in Durbin and Koopman (2001) chapter 5.

²¹Indeed, the AF/NS(2) model derivation was cross-checked using the full expression for the two-factor GATSM with constant market prices of risk from Chaplin (1987), and taking the limit of zero mean reversion for one of the factors.

(and hence two equivalent sets of estimated parameter), while the AF/NS(2) model has a unique maximum (and hence a unique estimated parameter set).²²

Second, the Level component for the AF/NS(2) model subsumes the constant parameter for the mean level of the short rate in the LF/GATSM. That again reduces by one the number of parameters to be estimated, but it more importantly allows both of the market prices of risk to be identified from an unrestricted estimation with a sample of zero-coupon data. As noted in Singleton (2006) pp. 342-343, the mean short rate parameter and the constant market prices of risk cannot all be identified for an LF/GATSM estimated with zero-coupon data. The suggested resolution is to set one of the constant market prices of risk to zero (albeit an arbitrary choice) or to use coupon-bearing data for the estimation (which is more complex).²³

Third, following the discussion in Collin-Dufresne et al. (2008), the state variables of the AF/NS(2) model have an economic meaning, in this case as the unique zeroth-order approximations to the persistent and non-persistent components of the generic GATSM. Together with the parameters, applications of the AF/NS(2) model can therefore be meaningfully compared across different countries and different sample periods. By contrast, the parameters and state variables obtained from an LF/GATSM cannot be compared from an economic perspective until a rotation to an economically meaningful representation is performed (although users often interpret the state variables approximately from the perspective of principal components).

One final econometric point is that the implicit econometric identification of the AF/NS(2) model uses $\theta_P = 0$, rather than $\theta_Q = 0$ as suggested in Singleton (2006) pp. 318-319 for the canonical two-LF/GATSM. The practical implication is to leave the constant market prices of risk in the measurement equation (i.e. $\sigma_1 \gamma_{0,1} \cdot \frac{1}{2} \tau$ in the interest rate function) and have no constant in the state equation. That identification is convenient for the empirical application in this article because it matches the implicit identification in RW. Conversely, the identification $\theta_Q = 0$ used for the AF/NS model in Christensen et al. (2010) eliminates the constant market prices of risk from the measurement equation and includes a constant in the state equation (which incorporates the constant market price of risk parameters). Nevertheless, both identifications provide observationally-equivalent representations of the data, and are within an invariant affine transformation of each other (see Singleton (2006) pp. 319-321).

6.2.2 Estimation results for the AF/NS(2) model

The Kalman filter recursion is used to evaluate the log likelihood for the model, and the latter is maximized numerically using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm.²⁴ Standard reparametrizations are employed to ensure trouble-free numerical evaluation; i.e. a Cholesky specification to ensure the covariance matrix always remains positive definite, and $\rho = \omega / (1 + |\omega|)$ to respect the ± 1 range for innovation correlations. The asymptotic standard errors are calculated using the Hessian matrix evaluated at the parameter values that maximize the likelihood function. Hamilton

²²In the terminology of Collin-Dufresne et al. (2008), the two-LF/GATSM is only locally identifiable and the AF/NS(2) model is globally identifiable.

²³Dai and Singleton (2000) footnote 14 discusses alternative restrictions that allow all the constant market prices of risk for GATSMs to be identified, but the point remains that some restriction is nevertheless required.

²⁴The function used is “fminunc” as supplied in the optimization toolbox of Matlab.

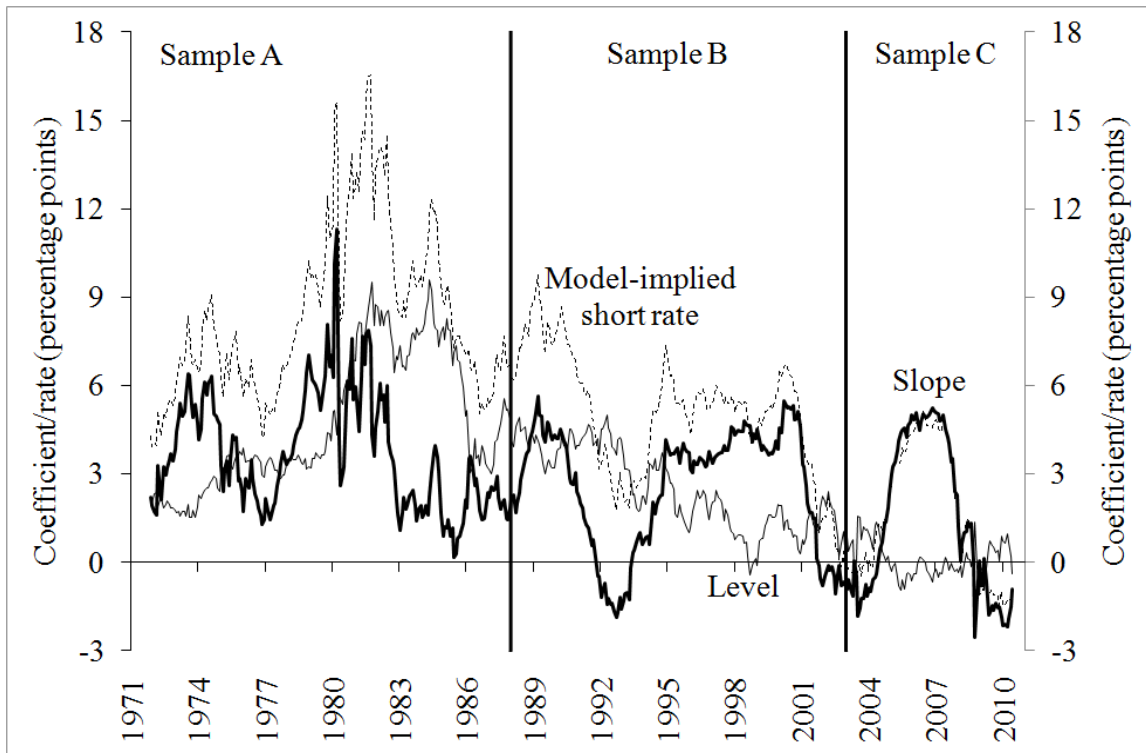


Figure 2: The estimated Level and Slope coefficients and the model-implied short rate, i.e. $\beta_1(t)$, $\beta_2(t)$, and $r(t) = \beta_1(t) + \beta_2(t)$, for the AF/NS(2) model estimated over the full sample using the 3-m/15-y data.

(1994) pp.139-142, 146-147, and 385-389 provides the appropriate background to all of these aspects.

Consistent with the discussion on identifiability from the previous subsection, convergence for the AF/NS(2) model estimation was timely and reliable, with no apparent sensitivity in the end result to the different starting values tested for the model parameters. Figure 2 illustrates the resulting state variables, i.e. the AF/NS(2) Level and Slope coefficients, for the estimation using the 3-month to 15-year yield curve data over the entire sample. With respect to their economic meaning, the AF/NS(2) Level and Slope coefficients respectively reflect the level and slope of the yield curve as represented by the 15-year rate and 3-month less 15-year spread in figure 1, and the model-implied short rate, i.e. $r(t) = \beta_1(t) + \beta_2(t)$, reflects the 3-month rate. The results for the individual sample periods and using the 3-m/30-y data are very similar to figure 2, and so are not separately reported. Note, however, that one aspect apparent in figure 2 is that $r(t)$ falls materially below zero in sample C. That in turn suggests that NS models (and by implication GATSM models) would be too simplistic for modeling the yield curve over sample C, particularly if it were important to strictly respect the zero bound for a given application.

Table 1 contains the estimated parameter values for the AF/NS(2) model using the 3-m/15-y data over the joint sample A+B and the individual samples A and B.²⁵ Table

²⁵The estimated parameters $\sigma_v^2(\tau_k)$ are not reported here and for the EA/AF/NS(2) model in the following section to save space. The typical values were respectively 0.10 and 0.16 percentage points

2 contains the estimated parameter values for the AF/NS(2) model using the 3-m/30-y data over the joint sample B+C and the individual samples B and C.

The first aspect of note is that the hypotheses of no change in the yield curve DGP between samples A and B, and samples B and C are soundly rejected, with a likelihood ratio statistics of 910.5 and 3532.4 respectively. Figure 2 illustrates that one of the main points of difference between the two samples is that the (time invariant) term premium function:

$$\text{TP}(\tau) = \sigma_1 \gamma_{0,1} \cdot \frac{1}{2} \tau + \sigma_2 \gamma_{0,2} \cdot \frac{1}{\phi} \left[1 - \frac{1}{\tau} F(\phi, \tau) \right] \quad (13)$$

is smaller in sample B than in sample A. That in turn mainly reflects a lower market price of risk for the Slope component, and also lower volatilities for the Level and Slope components. The term premium function falls again in sample C, mainly due to a further fall in the market price of risk for the Slope component.

Other points of note for the AF/NS(2) model estimates are the material decline in the mean-reversion parameter ϕ for the Slope coefficient from sample A to B, and the sign reversal of the innovation correlation parameter ρ from sample B to C. In practical terms, the latter suggests that positive innovations to the Slope coefficient (e.g. unanticipated policy tightenings) were on average associated with negative innovations to the Level coefficient (i.e. a fall in long-maturity yields) over sample C.

Parameter	Sample A+B		Sample A		Sample B	
ϕ	0.4994	(0.0083)	0.7028	(0.0140)	0.3931	(0.0082)
$\gamma_{0,1}$	0.1428	(0.0029)	0.1255	(0.0033)	0.1514	(0.0046)
$\gamma_{0,2}$	0.3079	(0.0130)	0.4659	(0.0223)	0.2223	(0.0169)
σ_1	0.0225	(0.0004)	0.0241	(0.0004)	0.0206	(0.0007)
σ_2	0.0339	(0.0014)	0.0367	(0.0018)	0.0295	(0.0021)
ρ	0.5729	(0.0307)	0.6728	(0.0427)	0.5257	(0.0403)
log L	17424.7		9706.7		8803.3	
$H_0:A=B$	910.5	[0.0000]				

Note: (·) standard errors, [·] χ^2 probabilities from LR statistic

for the 3-m/15-y and 3-m/30-y data.

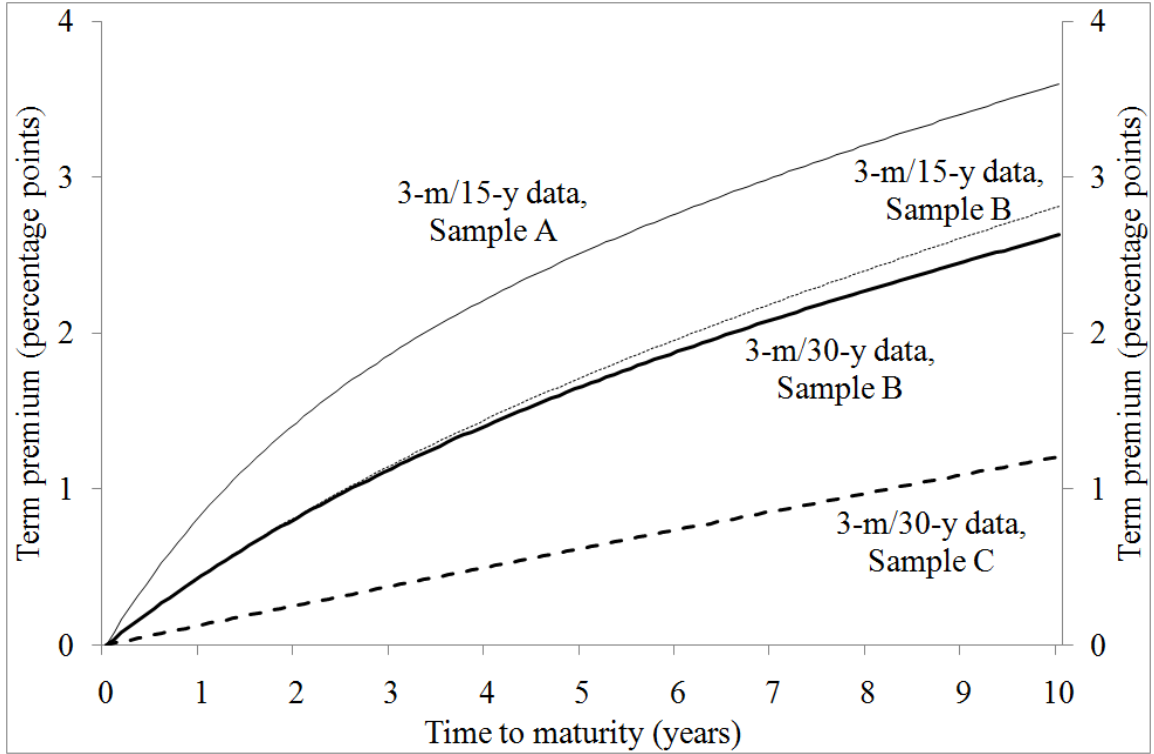


Figure 3: Term premium functions as implied by the point estimates of the AF/NS(2) model parameters in each sample, using the 3-m/15-y data and/or 3-m/30-y data as indicated.

Table 2:

Parameter estimates for the AF/NS(2) model with 3-m/30-y data

Parameter	Sample B+C		Sample B		Sample C	
ϕ	0.3733	(0.0059)	0.3884	(0.0069)	0.3355	(0.0078)
$\gamma_{0,1}$	0.1410	(0.0026)	0.1435	(0.0031)	0.1192	(0.0043)
$\gamma_{0,2}$	0.2392	(0.0114)	0.2895	(0.0151)	0.0158	(0.0183)
σ_1	0.0172	(0.0002)	0.0172	(0.0002)	0.0191	(0.0003)
σ_2	0.0238	(0.0011)	0.0250	(0.0013)	0.0213	(0.0015)
ρ	0.3136	(0.0381)	0.4098	(0.0380)	-0.3324	(0.0621)
log L	13621.6		10292.5		5095.4	
$H_0: B=C$	3532.4	[0.0000]				

Note: (·) standard errors, [·] χ^2 probabilities from LR statistic

6.3 The essentially affine two-factor AF/NS model

6.3.1 Model derivation

The AF/NS(2) model may readily be extended with an essentially affine specification for the market prices of risk; i.e. $\Gamma(t) = \gamma_0 + \gamma_1 \beta(t)$, where $\gamma_0 = [\gamma_{0,1}, \gamma_{0,2}]'$ and γ_1 is a 2×2 matrix of constants. Following Dai and Singleton (2002) appendix 1, the

measurement equation remains the same as for the AF/NS(2) model, and the state equation is modified by the matrix exponential $\exp(-\gamma_1)$ to give:

$$\begin{aligned}\beta(t + \Delta t) &= \Phi(\phi, \Delta t) \exp(-\gamma_1 \Delta t) \beta(t) + \varepsilon(t + \Delta t) \\ &= \exp(-\kappa \Delta t) \beta(t) + \varepsilon(t + \Delta t)\end{aligned}\quad (14)$$

where $\kappa = \text{diag}[0, \phi] + \gamma_1$.²⁶ Dai and Singleton (2002) also assumes that the eigenvalues of κ are strictly positive (which means the eigenvalues of $\exp(-\kappa \Delta t)$ will be less than 1 and so $\beta(t)$ will be stationary). That property is enforced in the estimation as follows.

From Higam (1996) p. 223 a non-symmetric matrix may be written as the sum of a symmetric matrix A_S and an antisymmetric matrix A_K , and A will be positive definite if A_S is positive definite. The latter can be generated with three additional parameters as $A_S = LL' + \text{diag}[0, \phi]$, where L is a 2×2 lower-diagonal matrix. The antisymmetric matrix can be generated with one additional parameter d and then setting $A_{12} = -A_{21} = d$ and $A_{11} = A_{22} = 0$. Directly calculating $\kappa = A_S + A_K$ and its eigenvalues gives:

$$\text{eig} \begin{bmatrix} a & b+d \\ b-d & c \end{bmatrix} = \frac{1}{2}c + \frac{1}{2}a \pm \frac{1}{2}\sqrt{(a-c)^2 - 4d^2 + 4b^2}\quad (15)$$

and a reparametrization:

$$d = \frac{e}{1 + |e|} \cdot \frac{1}{2}\sqrt{(a-c)^2 + 4b^2}\quad (16)$$

ensures that d will result in a positive value for the square root operand, therefore guaranteeing real positive eigenvalues for κ .²⁷

Equations 8 and 9 are a measurement and state equation that may be used in the Kalman filter to estimate the model. Regarding the additional elements required for Kalman filter, the covariance matrix for the state equation may be evaluated as:

$$\begin{aligned}Q &= \int_0^{\Delta t} \exp(-\kappa s) \Omega \exp(-\kappa' s) ds \\ &= V \begin{bmatrix} u_{11} \cdot F(2d_1, \Delta t) & u_{12} \cdot F(d_1 + d_2, \Delta t) \\ u_{21} \cdot F(d_1 + d_2, \Delta t) & u_{22} \cdot F(2d_2, \Delta t) \end{bmatrix} V'\end{aligned}\quad (17)$$

where VDV^{-1} is the eigensystem decomposition of κ , $D = \text{diag}[d_1, d_2]$, and the elements u_{ij} are those from $U = V^{-1}\Omega(V^{-1})'$.

The covariance matrix for the measurement equations is again assumed to be $H = \text{diag}[\sigma_v^2(\tau_1), \dots, \sigma_v^2(\tau_K)]$, and the starting values for the state variables and their

²⁶Like γ_0 , γ_1 has also been defined in this article to be positive.

²⁷This restriction could easily be modified if one wanted to allow for the possibility of a pair of complex conjugate eigenvalues with positive real parts. From an economic perspective, that would correspond to an expectation that innovations to the state variables would follow the product of a sinusoidal cycle and an exponential decay (rather than just an exponential decay) when returning to equilibrium.

covariance are respectively $\beta_{1|0} = [0, 0]'$ and

$$\begin{aligned} P_{1|0} &= \int_0^\infty \exp(-\kappa s) \Omega \exp(-\kappa' s) ds \\ &= V \begin{bmatrix} u_{11} \frac{1}{2d_1} & u_{12} \frac{1}{d_1+d_2} \\ u_{21} \frac{1}{d_1+d_2} & u_{22} \frac{1}{2d_2} \end{bmatrix} V' \end{aligned} \quad (18)$$

6.3.2 Estimation of the EA/AF/NS(2) model

Convergence for the model estimation via the Kalman filter was again timely and reliable, with no apparent sensitivity in the end result to the different starting values tested for the model parameters. All estimates of the Level and Slope coefficients for the EA/AF/NS(2) model were again very similar to figure 2 and so are not separately reported.

Tables 3 and 4 contain the estimated parameter values for the EA/AF/NS(2) model over the combined and individual samples. The hypotheses of no change in yield curve dynamics between the samples A and B, and the samples B and C are again soundly rejected by the likelihood ratio tests.

Figures 3 and 4 illustrate the term premium estimates for the time to maturity of five years for each of the different sample estimated. These are obtained by evaluating the following (now time-varying) function:

$$\text{TP}(t, \tau) = \text{TP}(\tau) + \left(\left[1, \frac{1}{\tau} F(\phi, \tau) \right] - [1, 1] [\kappa\tau]^{-1} [I - \exp(-\kappa\tau)] \right) \beta(t) \quad (19)$$

using the relevant estimated parameters and state variables with $\tau = 5$. Note that $\text{TP}(\tau)$ is the expression in equation 13.

Figures 3 and 4 confirm that changes in the average level of the term premium estimates are again a major point of difference between the individual samples, although the more complex specification for the term premium function in the EA/AF/NS(2) model makes it harder to attribute differences to individual parameters. Evaluating the point estimates of the time-invariant component $\text{TP}(\tau)$ of the term premium function for the EA/AF/NS(2) model confirms the AF/NS(2) pattern of results; i.e. a declining term premium from sample A to sample C. An exception is sample B based on 3-m-15-y data, where the EA/AF/NS(2) model estimates suggest a large time-invariant component, accompanied by a large range in the time-varying component $\text{TP}(t, \tau) - \text{TP}(\tau)$. However, the results using the 3-m/30-y data should in principle be superior given they exploit the additional information from longer maturities.

Regarding the time-varying components of the term premium estimates, sample B (based on 3-m/30-y data) shows the least variation in terms of the peak-to-trough range and the standard deviation of first differences. Sample C shows the most variation.

The significance of the time-varying component of the term premium estimates can be assessed with a likelihood ratio test given that the EA/AF/NS(2) model nests the AF/NS(2) model; i.e. the latter is the EA/AF/NS(2) model with $\gamma_1 = 0$ in $\Gamma(t) = \gamma_0 + \gamma_1 \beta(t)$. The likelihood ratio tests in tables 3 and 4 show that the estimates of γ_1 are typically highly significant. An exception is sample B using the 3-m/30-y data, suggesting that this period might be adequately represented for some applications with

time-invariant term premia.

Other points of note for the EA/AF/NS(2) model estimates are the confirmation of the material decline in the mean-reversion parameter ϕ and the sign reversal of the innovation correlation parameter ρ as already discussed for the AF/NS(2) model. Indeed, the model estimate suggests that the innovations become all but perfectly negative, albeit with an implausibly large confidence interval. That suggests some degree of overparametrization when applying the EA/AF/NS(2) model over the relatively short sample period.

Table 3:
Estimates for EA/AF/NS(2) model with 3-m/15-y data

Parameters	Sample A+B		Sample A		Sample B	
ϕ	0.5030	(0.0085)	0.7315	(0.0142)	0.3887	(0.0083)
$\gamma_{0,1}$	0.1355	(0.0123)	0.0914	(0.0122)	0.1923	(0.0056)
$\gamma_{0,2}$	0.2985	(0.0270)	0.5194	(0.0330)	0.1459	(0.0192)
σ_1	0.0227	(0.0006)	0.0193	(0.0006)	0.0276	(0.0007)
σ_2	0.0362	(0.0022)	0.0256	(0.0023)	0.0542	(0.0026)
ρ	0.3260	(0.2336)	0.1479	(0.5765)	0.4716	(0.1822)
$\gamma_{1,11}$	0.3081	(0.4400)	0.0002	(0.0326)	1.1247	(0.7635)
$\gamma_{1,12}$	-0.2610	(0.7073)	0.1892	(0.1555)	-0.0235	(0.6909)
$\gamma_{1,21}$	-0.3353	(0.4897)	-0.1925	(0.0673)	-0.0140	(3.7293)
$\gamma_{1,22}$	0.2920	(0.4770)	0.0152	(0.2650)	0.4468	(0.8513)
log L	17429.3		9106.5		8835.7	
$H_0:A=B$	1025.7	[0.0000]				
$H_0:\gamma_1=0$	9.3	[0.0548]	59.6	[0.0000]	64.8	[0.0000]

Note: (·) standard errors, [·] χ^2 probabilities from LR statistic

Table 4:
Estimates for EA/AF/NS(2) model with 3-m/30-y data

Parameters	Sample B+C		Sample B		Sample C	
ϕ	0.3319	(0.0053)	0.3904	(0.0068)	0.3177	(0.0078)
$\gamma_{0,1}$	0.1385	(0.0112)	0.1332	(0.0090)	0.1232	(0.0566)
$\gamma_{0,2}$	0.2314	(0.0307)	0.3014	(0.0266)	0.2383	(0.0593)
σ_1	0.0182	(0.0003)	0.0169	(0.0003)	0.0219	(0.0008)
σ_2	0.0236	(0.0008)	0.0240	(0.0007)	0.0217	(0.0053)
ρ	0.0317	(0.2371)	0.1695	(0.2201)	-0.9920	(1.5593)
$\gamma_{1,11}$	0.2776	(0.1245)	0.1739	(0.1823)	6.8665	(6.0884)
$\gamma_{1,12}$	-0.0516	(0.2160)	-0.0565	(0.3121)	0.8017	(1.8018)
$\gamma_{1,21}$	-0.2274	(0.1165)	-0.2182	(0.0980)	1.7737	(1.4409)
$\gamma_{1,22}$	0.0701	(0.1153)	0.1092	(0.1693)	0.2415	(0.3182)
log L	14982.3		10294.5		5180.2	
$H_0:B=C$	984.8	[0.0000]				
$H_0:\gamma_1=0$	2721.3	[0.0000]	4.0	[0.4066]	169.7	[0.0000]

Note: (·) standard errors, [·] χ^2 probabilities from LR statistic

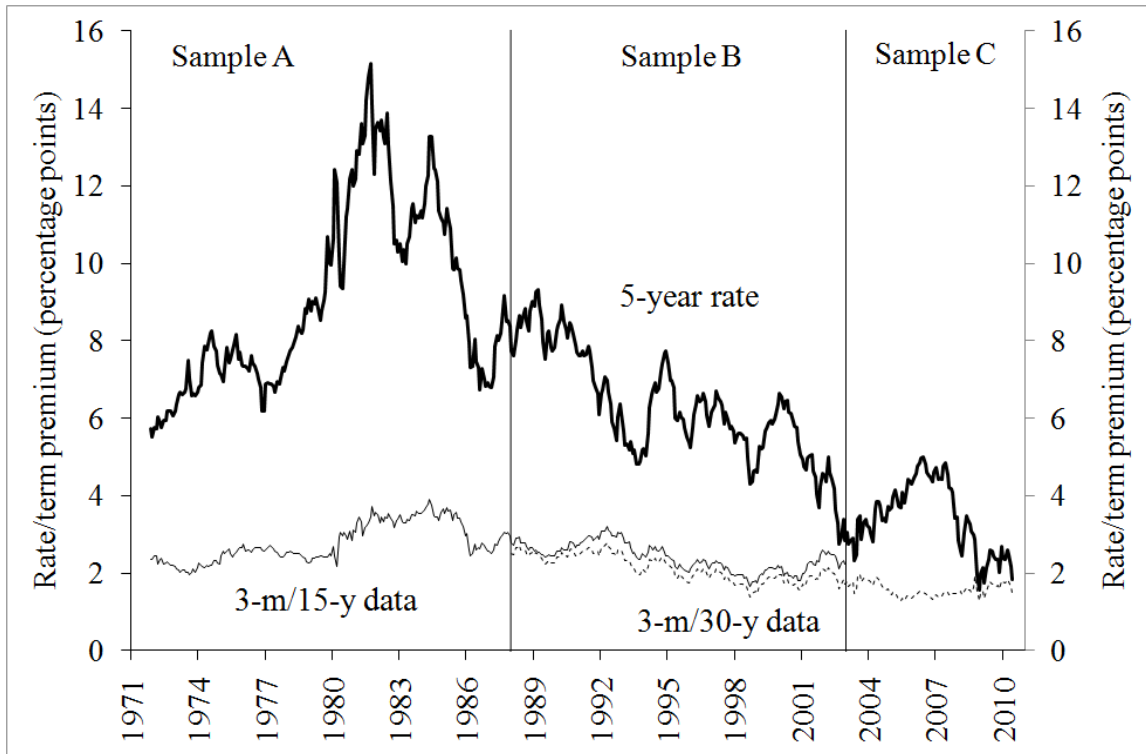


Figure 4: The 5-year interest rate data and the 5-year term premium implied by the point estimates of the EA/AF/NS(2) model parameters and state variables in each sample. Sample A+B uses the 3-m/15-y data and sample B+C uses 3-m/30-y data, as indicated.

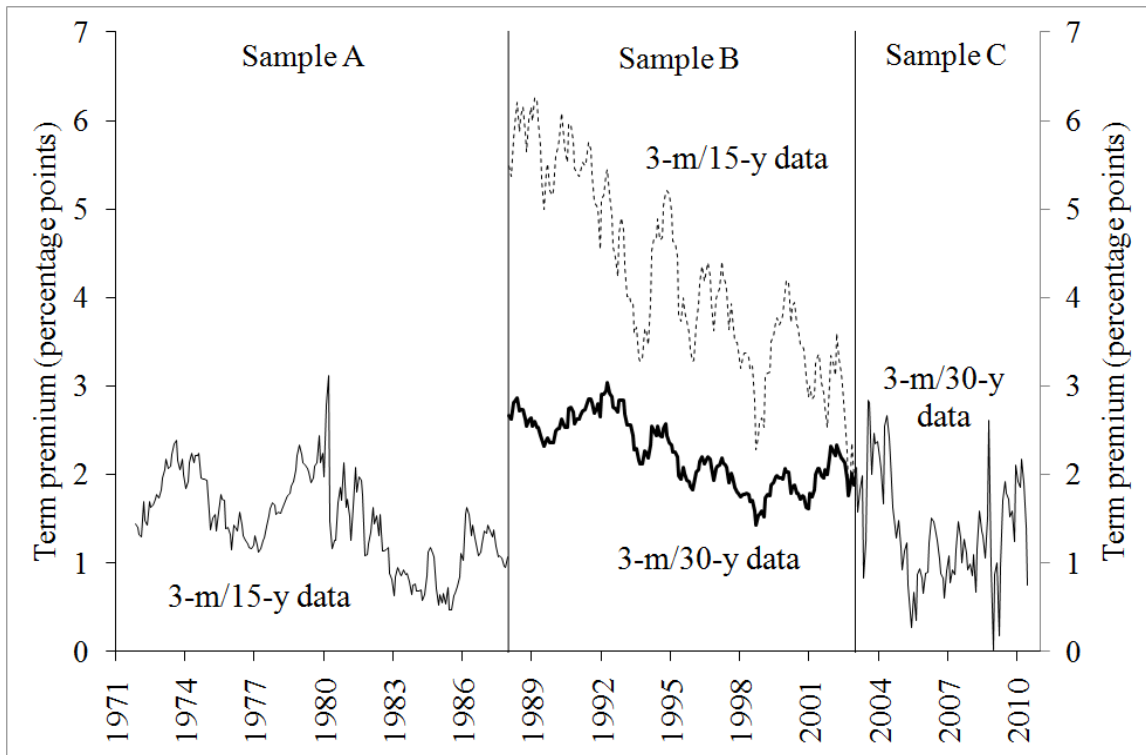


Figure 5: The 5-year term premia implied by the point estimates of the EA/AF/NS(2) model parameters and state variables in each sample, using the 3-m/15-y data and/or 3-m/30-y data as indicated.

When comparing the EA/AF/NS(2) model application to the RW two-LF/GATSM application, it is first worth noting that the model specification and the estimation approach differs in several respects. First, RW sets to zero the constant element of the market price of risk for the persistent component parameter, which allows the identification of the mean parameter for the short rate. Conversely, the EA/AF/NS(2) model allows all market price of risk parameters to be identified, and the results are material, statistically significant, and economically interesting.²⁸ Second, RW specifies a diagonal innovation covariance matrix for the state variables, while the EA/AF/NS(2) model allows for innovation correlations. These are found to be material and economically interesting, but not usually statistically significant. Third, RW restricts insignificant parameters for the remaining market price of risk specification (from an initial estimation) to zero, while the EA/AF/NS(2) model retains all of the market price of risk parameters.²⁹ The estimate of the γ_1 matrix for EA/AF/NS(2) model over the sample A+B provides a clear example of how the arbitrary initial zeroing of insignificant risk parameters could adversely affect the final model. That is, all of the individual estimates of the γ_1 matrix elements are insignificant, but γ_1 is significant as a whole.

Regarding the empirical results for the EA/AF/NS(2) model, they confirm the main finding in RW that a statistically significant change in term structure behavior occurred between samples A and B. The pattern of term premium estimates over the two samples is also similar to that of RW, with a fall in the variation of the 5-year term premium estimates from sample A to sample B. RW removes the sample averages from the term premium estimates, and so the absolute levels cannot be compared.

7 Conclusion

This article establishes that most NS models may be obtained as optimal approximations to the dynamic component of the generic GATSM outlined in Dai and Singleton (2002). That result provides a compelling case for applying NS models as standard tools for yield curve analysis in economics and finance: users get the well-established pragmatic benefits of NS models, including ease of estimation and a ready economic interpretation of their output, along with an assurance that the model is consistent with a well-accepted set of principles and assumptions for modeling the yield curve and its dynamics.

As a practical illustration of applying an NS model rather than a latent-factor GATSM, this article develops a two-factor arbitrage-free NS model and uses it to test for changes in U.S. yield dynamics. Estimating the NS model without parameter restrictions is reliable and timely, while the corresponding application of a two-factor latent GATSM in Rudebusch and Wu (2007) begins with various restrictions and imposes more as part of the estimation process. Nevertheless, the results from applying the NS model confirm the main findings of Rudebusch and Wu (2007): there was a very material change in the data-generating process for the U.S. yield curve between the

²⁸The EA/AF/NS(2) model also provides an estimate of the mean short rate; i.e. $\text{mean}[r(t)] = \text{mean}[\beta_1(t) + \beta_2(t)]$.

²⁹There are other differences that are not critical from an econometric perspective; i.e. RW assumes a lower-diagonal mean-reversion matrix for the state variables, and uses maximum-likelihood estimation assuming no measurement errors for the interest rate data of two selected maturities. The maturity span of the data is also different, from 1-month to 5-years.

sample from 1971 to 1988 and the sample from 1988 to 2002. An additional estimation of the NS model using data from 2003 to 2010 also indicates a further material change in U.S. yield curve behavior.

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A CIR dynamics

This appendix shows by example that dynamic term structure models with Cox et al. (1985b)/square-root innovations cannot be optimally approximated using NS factor loadings, in the sense of following a procedure analogous to the exposition in section 3.

Assume N independent factors each with the form $dX_n(t) = \kappa_n [\theta_n - X_n(t)] dt + \sigma_n \sqrt{X_n(t)} dW(t)$ under the risk-neutral Q measure.

Then $P(t, T) = \exp \left[\sum_{n=1}^N A_n(t, T) + B_n(t, T) X_n(t) \right]$ where each $B_n(t, T)$ has the standard Cox et al. (1985b) form:³⁰

$$B_n(t, T) = \frac{2[1 - \exp(\gamma_n \tau)]}{(\gamma_n + \kappa_n)[\exp(\gamma_n \tau) - 1] + 2\gamma_n} \quad (20)$$

with $\gamma_n = \sqrt{\kappa_n^2 + 2\sigma_n^2}$.

The associated forward rate curve is:

$$f(t, T) = a_0 + \sum_{n=1}^N \frac{4\gamma_n^2 \exp(\gamma_n \tau)}{[(\gamma_n + \kappa_n)[\exp(\gamma_n \tau) - 1] + 2\gamma_n]^2} X_n(t) - \frac{\partial}{\partial \tau} A_n(\tau) \quad (21)$$

The relative complexity of this functional form of maturity means that a central exponential decay term $\exp(-\phi\tau)$ cannot be factored out of each factor loading as for the Gaussian case in section 3.

This incompatibility of the NS class of yield curve models with CIR/square-root dynamics is unfortunate, because CIR models have the well-known advantage over GATSMs of respecting the zero bound for interest rates. One resulting implication is that, in cases where the probability of zero interest rates from Gaussian dynamics is material, a non-Gaussian dynamic term structure model might be more appropriate than an NS model. That caveat applies in particular if the application requires the zero bound to be strictly respected (e.g. for financial market applications such as option pricing).

³⁰See, for example, Hull (2000) p. 570. The associated $A_n(t, T)$ terms have the form $-a_0\tau + A^*(\tau)$, and so have no influence on the factor loadings.

B Details of calculations for section 6

B.1 AF/NS(2) forward rate curve

B.1.1 AF/NS(2) forward rate expression

Equation 5 from the main text expresses the AF/NS(2) forward rate in terms of three components: (1) the expected path of the short rate component, the term premium component, and the volatility effect component. These are evaluated respectively in the subsections below.

B.1.2 The expected path of the short rate component

$$\begin{aligned}\mathbb{E}_t[r(t+\tau)] &= \beta_1(t) + \beta_2(t) \cdot \exp(-\phi\tau) \\ &= [1, \exp(-\phi\tau)] \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \end{bmatrix} \\ &= b(\tau) \beta(t)\end{aligned}$$

where $b(\tau) = [1, \exp(-\phi\tau)]$ and $\beta(t) = [\beta_1(t), \beta_2(t)]'$.

B.1.3 Term premium component

$$\int_0^\tau [\sigma_1, \sigma_2 \exp(-\phi s)] \begin{bmatrix} \gamma_{0,1} \\ \gamma_{0,2} \end{bmatrix} ds = \sigma_1 \gamma_{0,1} \int_0^\tau ds + \sigma_2 \gamma_{0,2} \int_0^\tau \exp(-\phi s) ds$$

The Level and Slope term premium components are evaluated respectively in the subsections below.

Term premium component for Level

$$\begin{aligned}\sigma_1 \gamma_{0,1} \int_0^\tau ds &= \sigma_1 \gamma_{0,1} (s|_0^\tau) \\ &= \sigma_1 \gamma_{0,1} \cdot \tau\end{aligned}$$

Term premium component for Slope

$$\begin{aligned}\int_0^\tau \sigma_2 \gamma_{0,2} \exp(-\phi[\tau-s]) ds &= \sigma_2 \gamma_{0,2} \frac{1}{\phi} \exp(-\phi[\tau-s]) \Big|_0^\tau \\ &= \sigma_2 \gamma_{0,2} \cdot \frac{1}{\phi} [1 - \exp(-\phi\tau)] \\ &= \sigma_2 \gamma_{0,2} \cdot F(\phi, \tau)\end{aligned}$$

where $F(\phi, \tau) = \frac{1}{\phi} [1 - \exp(-\phi\tau)]$.

B.1.4 Volatility effect component

$$\begin{aligned}
& \int_0^\tau [\sigma_1, \sigma_2 \exp(-\phi[\tau-s])] \left(\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \int_s^\tau \begin{bmatrix} \sigma_1 \\ \sigma_2 \exp(-\phi[u-s]) \end{bmatrix} du \right) ds \\
= & \int_0^\tau [\sigma_1, \sigma_2 \exp(-\phi[\tau-s])] \begin{bmatrix} \sigma_1 \int_s^\tau du + \rho \sigma_2 \int_s^\tau \exp(-\phi[u-s]) du \\ \rho \sigma_1 \int_s^\tau du + \sigma_2 \int_s^\tau \exp(-\phi[u-s]) du \end{bmatrix} ds \\
= & \sigma_1^2 \int_0^\tau \left(\int_s^\tau du \right) ds \\
& + \rho \sigma_1 \sigma_2 \int_0^\tau \left(\int_s^\tau \exp(-\phi[u-s]) du \right) ds \\
& + \rho \sigma_1 \sigma_2 \int_0^\tau \exp(-\phi[\tau-s]) \left(\int_s^\tau du \right) ds \\
& + \sigma_2^2 \int_0^\tau \exp(-\phi[\tau-s]) \left(\int_s^\tau \exp(-\phi[u-s]) du \right) ds
\end{aligned}$$

where the last four lines are, respectively, the Level/Level, Level/Slope, Slope/Level, and Slope/Slope components. These are evaluated in turn in the subsections below.

Volatility effect component for Level/Level

$$\begin{aligned}
\int_0^\tau \left(\int_s^\tau du \right) ds &= \int_0^\tau (u|_s^\tau) ds \\
&= \int_0^\tau (\tau - s) ds \\
&= \left(\tau s - \frac{1}{2} s^2 \right) \Big|_0^\tau \\
&= \frac{1}{2} \tau^2
\end{aligned}$$

Volatility effect component for Level/Slope

$$\begin{aligned}
\int_0^\tau \left(\int_s^\tau \exp(-\phi[u-s]) du \right) ds &= \int_0^\tau \left(-\frac{1}{\phi} \exp(-\phi[u-s]) \Big|_s^\tau \right) ds \\
&= \frac{1}{\phi} \int_0^\tau (1 - \exp(-\phi[\tau-s])) ds \\
&= \frac{1}{\phi} \left(s - \frac{1}{\phi} \exp(-\phi[\tau-s]) \right) \Big|_0^\tau \\
&= \frac{\tau}{\phi} - \frac{1}{\phi^2} + \frac{1}{\phi^2} \exp(-\phi\tau)
\end{aligned}$$

Volatility effect component for Slope/Level

$$\begin{aligned}\int_0^\tau \exp(-\phi[\tau-s]) \left(\int_s^\tau du \right) ds &= \int_0^\tau \exp(-\phi[\tau-s]) (\tau-s) ds \\ &= \left(\exp(-\phi[\tau-s]) \frac{1}{\phi^2} (1 + \tau\phi - s\phi) \right) \Big|_0^\tau \\ &= \frac{1}{\phi^2} - \frac{1}{\phi^2} \exp(-\phi\tau) - \frac{\tau}{\phi} \exp(-\phi\tau)\end{aligned}$$

Volatility effect component for Level/Slope + Slope/Level

$$\begin{aligned}\int_0^\tau \left(\int_s^\tau \exp(-\phi[u-s]) du \right) ds + \int_0^\tau \exp(-\phi[\tau-s]) \left(\int_s^\tau du \right) ds &= \frac{\tau}{\phi} - \frac{\tau}{\phi} \exp(-\phi\tau) \\ &= \tau \cdot F(\phi, \tau)\end{aligned}$$

Volatility effect component for Slope/Slope

$$\begin{aligned}&\int_0^\tau \exp(-\phi[\tau-s]) \left(\int_s^\tau \exp(-\phi[u-s]) du \right) ds \\ &= \int_0^\tau \exp(-\phi s) \left(\frac{1}{\phi} (\exp(-\phi[\tau-s]) - 1) \right) ds \\ &= \frac{1}{\phi^2} \left(\exp(2\phi\tau) \left(\exp(-\phi[\tau-s]) - \frac{1}{2} \exp(-2\phi s) \right) \right) \Big|_0^\tau \\ &= \frac{1}{2\phi^2} [1 - \exp(-\phi\tau)]^2\end{aligned}$$

B.1.5 AF/NS(2) forward rate curve

Substituting the results above into the AF/NS(2) forward rate expression gives equation 6.

B.2 AF/NS(2) interest rate curve

The AF/NS(2) interest rate curve may be evaluated using the standard relationship $R(t, \tau) = \frac{1}{\tau} \int_0^\tau f(t, \tau) d\tau$ for each component of the AF/NS(2) forward rate curve. The calculations are undertaken respectively in the subsections below.

B.2.1 Short rate Level component

$$\begin{aligned}\frac{1}{\tau} \int_0^\tau d\tau &= \frac{1}{\tau} (\tau|_0^\tau) \\ &= \frac{1}{\tau} (\tau) \\ &= 1\end{aligned}$$

B.2.2 Short rate Slope component

$$\begin{aligned}\frac{1}{\tau} \int_0^\tau \exp(-\phi\tau) d\tau &= \frac{1}{\tau} \left(-\frac{1}{\phi} \exp(-\phi\tau) \Big|_0^\tau \right) \\ &= \frac{1}{\tau} \left(\frac{1}{\phi} [1 - \exp(-\phi\tau)] \right) \\ &= \frac{1}{\tau} F(\phi, \tau)\end{aligned}$$

B.2.3 Term premium Level component

$$\begin{aligned}\frac{1}{\tau} \int_0^\tau \tau d\tau &= \frac{1}{\tau} \left(\frac{1}{2} \tau^2 \Big|_0^\tau \right) \\ &= \frac{1}{\tau} \left(\frac{1}{2} \tau^2 \right) \\ &= \frac{1}{2}\end{aligned}$$

B.2.4 Term premium Slope component

$$\begin{aligned}\frac{1}{\tau} \int_0^\tau F(\phi, \tau) d\tau &= \frac{1}{\tau} \int_0^\tau \frac{1}{\phi} [1 - \exp(-\phi\tau)] d\tau \\ &= \frac{1}{\tau} \left(\frac{\tau}{\phi} + \frac{1}{\phi^2} \exp(-\phi\tau) \Big|_0^\tau \right) \\ &= \frac{1}{\tau} \left[\frac{\tau}{\phi} - \frac{1}{\phi^2} + \frac{1}{\phi^2} \exp(-\phi\tau) \right] \\ &= \frac{1}{\phi} \left[1 - \frac{1}{\tau} F(\phi, \tau) \right]\end{aligned}$$

B.2.5 Volatility effect Level/Level component

$$\begin{aligned}\frac{1}{\tau} \int_0^\tau \frac{1}{2} \tau^2 d\tau &= \frac{1}{\tau} \left(\frac{1}{6} \tau^3 \Big|_0^\tau \right) \\ &= \frac{1}{\tau} \left(\frac{1}{6} \tau^3 \right) \\ &= \frac{1}{6}\end{aligned}$$

B.2.6 Volatility effect Slope/Slope component

$$\begin{aligned}
\frac{1}{\tau} \int_0^\tau [F(\phi, \tau)]^2 d\tau &= \frac{1}{\tau} \int_0^\tau \frac{1}{\phi^2} [1 - \exp(-\phi\tau)]^2 d\tau \\
&= \frac{1}{\tau} \left(\frac{1}{2\phi^2} \tau + \frac{1}{\phi^3} \exp(-\phi\tau) - \frac{1}{4\phi^3} \exp(-2\phi\tau) \Big|_0^\tau \right) \\
&= \frac{1}{\tau} \left[\frac{\tau}{2\phi^2} - \frac{3}{4\phi^3} + \frac{1}{\phi^3} \exp(-\phi\tau) - \frac{1}{4\phi^3} \exp(-2\phi\tau) \right] \\
&= \frac{1}{2\phi^2} - \frac{1}{2\phi^3\tau} [1 - \exp(-\phi\tau)] - \frac{1}{4\phi^3\tau} [1 - \exp(-\phi\tau)]^2 \\
&= \frac{1}{2\phi^2} \left(1 - \frac{1}{\tau} F(\phi, \tau) - \frac{1}{2\tau} \phi [F(\phi, \tau)]^2 \right)
\end{aligned}$$

B.2.7 Volatility effect Level/Slope + Slope/Level component

$$\begin{aligned}
\frac{1}{\tau} \int_0^\tau \tau F(\phi, \tau) d\tau &= \frac{1}{\tau} \int_0^\tau \frac{\tau}{\phi} [1 - \exp(-\phi\tau)] d\tau \\
&= \frac{1}{\tau} \left(\frac{1}{2\phi} \tau^2 + \frac{1}{\phi^3} \exp(-\phi\tau) + \frac{\tau}{\phi^2} \exp(-\phi\tau) \Big|_0^\tau \right) \\
&= \frac{1}{\tau} \left(\frac{1}{2\phi} \tau^2 - \frac{1}{\phi^3} + \frac{1}{\phi^3} \exp(-\phi\tau) + \frac{\tau}{\phi^2} \exp(-\phi\tau) \right) \\
&= \frac{1}{2\phi} \tau - \frac{1}{\phi^2\tau} F(\phi, \tau) + \frac{1}{\phi^2} - \frac{1}{\phi^2} \exp(-\phi\tau) \\
&= \frac{1}{\phi^2} \left(1 - \frac{1}{\tau} F(\phi, \tau) + \frac{1}{2} \phi\tau - \frac{1}{\phi^2\tau} F(\phi, \tau) \right)
\end{aligned}$$

B.2.8 Final interest rate curve expression

$$\begin{aligned}
R(t, \tau) &= \beta_1(t) + \beta_2(t) \cdot \frac{1}{\tau} F(\phi, \tau) \\
&\quad + \sigma_1 \gamma_1 \cdot \frac{1}{2} \tau + \sigma_2 \gamma_2 \cdot \frac{1}{\phi} \left[1 - \frac{1}{\tau} F(\phi, \tau) \right] \\
&\quad - \sigma_1^2 \cdot \frac{1}{6} \tau^2 - \sigma_2^2 \cdot \frac{1}{2\phi^2} \left[1 - \frac{1}{\tau} F(\phi, \tau) - \frac{1}{2\tau} \phi [F(\phi, \tau)]^2 \right] \\
&\quad - \rho \sigma_1 \sigma_2 \cdot \frac{1}{\phi^2} \left[1 - \frac{1}{\tau} F(\phi, \tau) + \frac{1}{2} \phi\tau - \frac{1}{\phi^2\tau} F(\phi, \tau) \right]
\end{aligned}$$

which may be expressed in the form $R(t, \tau) = \bar{a}(\tau) + \bar{b}(\tau) \beta(t)$ as in the main text.

B.3 AF/NS(2) model Kalman filter calculations

B.3.1 State equation

The AF/NS(2) state equation may be derived directly from the expected path of the short rate, i.e.:

$$\begin{aligned}
\mathbb{E}_t \{ \mathbb{E}_{t+\tau} r(t + \tau + \Delta t) \} &= \mathbb{E}_t [r(t + \tau + \Delta t)] \\
\mathbb{E}_t \{ [1, \exp(-\phi\tau)] \beta(t + \Delta t) \} &= [1, \exp(-\phi[\tau + \Delta t])] \beta(t) \\
[1, \exp(-\phi\tau)] \mathbb{E}_t \{ \beta(t + \Delta t) \} &= [1, \exp(-\phi\tau)] \begin{bmatrix} 1 & 0 \\ 0 & \exp(-\phi\Delta t) \end{bmatrix} \beta(t) \\
\mathbb{E}_t \{ \beta(t + \Delta t) \} &= \Phi(\phi, \Delta t) \beta(t)
\end{aligned}$$

Removing the expectations operator, the state equation is therefore:

$$\beta(t + \Delta t) = \Phi(\phi, \Delta t) \beta(t) + \varepsilon(t + \Delta t)$$

as in equation 8, where $\mathbb{E}_t [\varepsilon(t + \Delta t)] = 0$. Note that $\varepsilon(t + \Delta t)$ has a correlated bivariate Gaussian distribution given the correlated innovations assumed to underlie equation 5, i.e. $\Omega [dW_1(t), \exp(-\phi\tau) \cdot dW_2(t)]$.

B.3.2 State covariance matrix

$$\begin{aligned}
Q &= \int_0^{\Delta t} \Phi(\phi, s) \Omega [\Phi(\phi, s)]' ds \\
&= \int_0^{\Delta t} \begin{bmatrix} 1 & 0 \\ 0 & \exp(-\phi s) \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \exp(-\phi s) \end{bmatrix} ds \\
&= \int_0^{\Delta t} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \exp(-\phi s) \\ \rho\sigma_1\sigma_2 \exp(-\phi s) & \sigma_2^2 \exp(-2\phi s) \end{bmatrix} ds \\
&= \left[\begin{array}{cc} \sigma_1^2 s & -\rho\sigma_1\sigma_2 \frac{1}{\phi} \exp(-\phi s) \\ -\rho\sigma_1\sigma_2 \exp(-\phi s) & -\sigma_2^2 \frac{1}{2\phi} \exp(-2\phi s) \end{array} \right] \Big|_0^{\Delta t} \\
&= \begin{bmatrix} \sigma_1^2 \Delta t & \rho\sigma_1\sigma_2 \left[\frac{1}{\phi} - \frac{1}{\phi} \exp(-\phi \Delta t) \right] \\ \rho\sigma_1\sigma_2 \left[\frac{1}{\phi} - \frac{1}{\phi} \exp(-\phi \Delta t) \right] & \sigma_2^2 \frac{1}{2\phi} [1 - \exp(-2\phi \Delta t)] \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1^2 \cdot \Delta t & \rho\sigma_1\sigma_2 \cdot F(\phi, \Delta t) \\ \rho\sigma_1\sigma_2 \cdot F(\phi, \Delta t) & \sigma_2^2 \cdot F(2\phi, \Delta t) \end{bmatrix}
\end{aligned}$$

B.3.3 Unconditional state covariance matrix

$$\begin{aligned}
P_{1|0} &= \int_0^{\infty} \Phi(\phi, s) \Omega [\Phi(\phi, s)]' ds \\
&= \left[\begin{array}{cc} \sigma_1^2 s & -\rho\sigma_1\sigma_2 \frac{1}{\phi} \exp(-\phi s) \\ -\rho\sigma_1\sigma_2 \exp(-\phi s) & -\sigma_2^2 \frac{1}{2\phi} \exp(-2\phi s) \end{array} \right] \Big|_0^{\infty} \\
&= \begin{bmatrix} \text{undef} & \rho\sigma_1\sigma_2 \frac{1}{\phi} \\ \rho\sigma_1\sigma_2 \frac{1}{\phi} & \sigma_2^2 \frac{1}{2\phi} \end{bmatrix}
\end{aligned}$$

where the indefinite integral evaluations use the results from the previous subsection. Note that $P_{1|0}(1, 1) = \sigma_\eta^2 / (1 - \psi^2)$ replaces the undefined expression “undef” to give $P_{1|0}$ as in equation 12.

B.3.4 Cholesky specification for Ω

$$\begin{aligned} LL' &= \Omega \\ \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} \sigma_1 & \rho\sigma_2 \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix} &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \rho^2\sigma_2^2 + \sigma_2^2(-\rho^2 + 1) \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \end{aligned}$$

B.4 AF/NS(2) term premium functions

The AF/NS(2) forward term premium function can be obtained by subtracting the AF/NS(2) forward rate expression excluding term premia (i.e. by setting $\gamma_0 = 0$) from the AF/NS(2) forward rate expression with term premia, i.e.:

$$\begin{aligned} \text{FTP}(\tau) &= f(t, \tau) - [f(t, \tau) | \gamma_0 = 0] \\ &= \sigma_1\gamma_{0,1} \cdot \tau + \sigma_2\gamma_{0,2} \cdot F(\phi, \tau) \end{aligned}$$

The interest rate term premium function is then:

$$\begin{aligned} \text{TP}(\tau) &= \frac{1}{\tau} \int_0^\tau \text{FTP}(\tau) d\tau \\ &= \sigma_1\gamma_1 \cdot \frac{1}{2}\tau + \sigma_2\gamma_2 \cdot \frac{1}{\phi} \left[1 - \frac{1}{\tau} F(\phi, \tau) \right] \end{aligned}$$

B.5 The EA/AF/NS(2) model

There is almost certainly a straightforward way to establish the expressions for the EA/AF/NS(2) model using a suitable modification to the HJM framework. But, assuming that route exists, it has unfortunately escaped the investigations of the author.

However, it turns out that the EA/AF/NS(2) model is precisely a special case of the Dai and Singleton (2002, hereafter DS) generic GATSM, as introduced in section 2, with two factors and limit of zero for one of the rates of mean reversion. The derivations in the following subsections are therefore based on the generic GATSM with the substitution of the EA/AF/NS(2) model parameters; i.e. $X(t) = \beta(t)$, $\xi_0 = 0$, $\xi_1 = [1, 1]'$, $K_Q = \text{diag}[0, \phi]$, $\pi_0 = -\gamma_0$, $\pi_1 = -\gamma_1$, and $K_P = \kappa = \text{diag}[0, \phi] + \gamma_1$.

Regarding the EA/AF/NS(2) forward rate and interest rate curves, the time-invariant term premia associated with the constant market prices of risk γ_0 remain via the identification $\theta_P = 0$, as discussed at the end of section 6.2.1. The time-varying term premia associated with matrix γ_1 are captured in the state equation, as discussed in the following subsection. The measurement equation for the EA/AF/NS(2) model therefore remains identical to that of the AF/NS(2) model.

B.6 EA/AF/NS(2) model Kalman filter calculations

B.6.1 State equation

DS appendix A gives the following expression for the generic GATSM state equation:

$$\mathbb{E}_t [X(t + \Delta t)] = \exp(-K_P \tau) X(t) + [I - \exp(-K_P \tau)] \theta_P$$

Substituting the EA/AF/NS(2) parameters from the previous subsection into the generic GATSM state equation expression gives $\exp(-K_Q \Delta t) = \text{diag}[1, \exp(-\phi \Delta t)] = \Phi(\phi, \Delta t)$, $\exp(\pi_1 \tau) = \exp(-\gamma_1 \tau)$, and $\exp(-K_P \Delta t) = \exp(-\kappa \Delta t)$. Therefore the EA/AF/NS(2) state equation is:

$$\begin{aligned} \mathbb{E}_t [\beta(t + \Delta t)] &= \Phi(\phi, \Delta t) \exp(-\gamma_1 \tau) \beta(t) \\ &= \exp(-\kappa \tau) \beta(t) \end{aligned}$$

as in equation 14. Note that $\varepsilon(t + \Delta t)$ is Gaussian as for the AF/NS(2) model.

B.6.2 State covariance matrix

$$\begin{aligned} Q &= \int_0^{\Delta t} \exp(-\kappa s) \Omega \exp(-\kappa' s) ds \\ &= \int_0^{\Delta t} \exp(-VDV^{-1}s) \Omega [\exp(-VDV^{-1}s)]' ds \\ &= \int_0^{\Delta t} V \exp(-Ds) V^{-1} \Omega [\exp(-VDV^{-1}s)]' ds \\ &= \int_0^{\Delta t} V \begin{bmatrix} \exp(-d_1 s) & 0 \\ 0 & \exp(-d_2 s) \end{bmatrix} V^{-1} \Omega (V^{-1})' \begin{bmatrix} \exp(-d_1 s) & 0 \\ 0 & \exp(-d_2 s) \end{bmatrix} V' ds \\ &= V \left(\int_0^{\Delta t} \begin{bmatrix} \exp(-d_1 s) & 0 \\ 0 & \exp(-d_2 s) \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} \exp(-d_1 s) & 0 \\ 0 & \exp(-d_2 s) \end{bmatrix} ds \right) V' \\ &= V \left(\int_0^{\Delta t} \begin{bmatrix} u_{11} \exp(-2d_1 s) & u_{12} \exp(-[d_1 + d_2] s) \\ u_{21} \exp(-[d_1 + d_2] s) & u_{22} \exp(-2d_2 s) \end{bmatrix} ds \right) V' \\ &= V \begin{bmatrix} u_{11} \cdot F(2d_1, \Delta t) & u_{12} \cdot F(d_1 + d_2, \Delta t) \\ u_{21} \cdot F(d_1 + d_2, \Delta t) & u_{22} \cdot F(2d_2, \Delta t) \end{bmatrix} V' \end{aligned}$$

B.6.3 Unconditional state covariance matrix

$$\begin{aligned} P_{1|0} &= \int_0^{\infty} \exp(-\kappa s) \Omega \exp(-\kappa' s) ds \\ &= V \begin{bmatrix} u_{11} F_2(2d_1, \infty) & u_{12} F_2(d_1 + d_2, \infty) \\ u_{21} F_2(d_1 + d_2, \infty) & u_{22} F_2(2d_2, \infty) \end{bmatrix} V' \\ &= V \begin{bmatrix} \frac{u_{11}}{2d_1} & \frac{u_{12}}{d_1 + d_2} \\ \frac{u_{21}}{d_1 + d_2} & \frac{u_{22}}{2d_2} \end{bmatrix} V' \end{aligned}$$

where the integral evaluations use the results from the previous subsection.

B.7 EA/AF/NS(2) term premium functions

The time-varying component of the EA/AF/NS(2) forward term premium function may be obtained using the time-varying component of the DS forward term premium expression for the generic GATSM model.

B.7.1 DS discrete-time expression for forward term premium

DS appendix A obtains the following generic GATSM forward term premium expression p_t^n using discrete increments of time Δt :

$$p_t^n = f_t^n - \mathbb{E}_t[r_{t+n\Delta t}]$$

where:

$$f_t^n = -\frac{1}{\Delta} \log \frac{P(t, [n+1]\Delta t)}{P(t, n\Delta t)}$$

and:

$$\begin{aligned} \mathbb{E}_t[r_{t+n\Delta t}] &= \mu_n + \nu_n' X(t) \\ \mu_n &= a_1 + \theta_P' [I - \exp(-K_P' n\Delta t)] b_1 \\ \nu_n &= \exp(-K_P' n\Delta t) b_1 \end{aligned}$$

Note that a_1 and b_1 are constants associated with the one-period interest rate $r_t = a_1 + b_1 X(t)$. DS also notes that f_t^n is the one-period forward deliverable n -periods forward, and $\mathbb{E}_t[r_{t+n\Delta t}]$ is the conditional mean of the short rate.

B.7.2 DS continuous-time expression for forward term premium

In the reverse order of which they were introduced, take the limit of each quantity in the previous section as $\Delta t \rightarrow 0$. Hence, $\lim_{\Delta t \rightarrow 0} r_t = r(t) = \xi_0 + \xi_1' X(t)$ as defined in section 2, so $\lim_{\Delta t \rightarrow 0} b_1 = \xi_1$ and $\lim_{\Delta t \rightarrow 0} a_1 = \xi_0$. Therefore:

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \nu_n &= \nu(\tau) = \exp(-K_P' \tau) \xi_1 \\ \lim_{\Delta t \rightarrow 0} \mu_n &= \mu(\tau) = \xi_0 + \theta_P' [I - \exp(-K_P' \tau)] \xi_0 \\ \lim_{\Delta t \rightarrow 0} \mathbb{E}_t[r_{t+n\Delta t}] &= \xi_0 + \theta_P' [I - \exp(-K_P' \tau)] \xi_0 + \xi_1' \exp(-K_P \tau) X(t) \end{aligned}$$

Regarding the forward rate:

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} f_t^n &= -\frac{\partial}{\partial T} \log P(t, T) = f(t, T) \\ &= \xi_0 + [\exp(-K_Q' \tau) \xi_1]' X(t) - \frac{\partial}{\partial \tau} A(\tau) \end{aligned}$$

where $P(t, T)$ and $f(t, T)$ are as outlined in section 2.

The forward term premium expression therefore becomes:

$$\begin{aligned}
\lim_{\Delta t \rightarrow 0} p_t^n &= p(t, \tau) \\
&= f(t, T) - \mathbb{E}_t[r_{t+n\Delta t}] \\
&= \xi_0 + \xi_1' \exp(-K_Q \tau) X(t) - \frac{\partial}{\partial \tau} A(\tau) \\
&\quad - \xi_0 - \theta'_P [I - \exp(-K'_P \tau)] \xi_0 - \xi_1' \exp(-K_P \tau) X(t)
\end{aligned}$$

and the time-varying component of $p(t, \tau)$ is:

$$p(t, \tau) - p(\tau) = \xi_1' \exp(-K_Q \tau) X(t) - \xi_1' \exp(-K_P \tau) X(t)$$

B.7.3 EA/AF/NS(2) forward term premium function

Substituting the EA/AF/NS(2) parameters and expressions from section B.5 and B.6.1 into the time-varying component of $p(t, \tau)$ from the previous section gives:

$$\begin{aligned}
p(t, \tau) - p(\tau) &= \{[1, 1] \Phi(\phi, \Delta t)\} \beta(t) - [1, 1] \exp(-\kappa \tau) \beta(t) \\
\text{FTP}(t, \tau) - \text{FTP}(\tau) &= [1, \exp(-\phi \Delta t)] \beta(t) - [1, 1] \exp(-\kappa \tau) \beta(t)
\end{aligned}$$

B.7.4 EA/AF/NS(2) interest rate term premium function

$$\begin{aligned}
\text{TP}(t, \tau) &= \frac{1}{\tau} \int_0^\tau \text{FTP}(t, \tau) d\tau \\
&= \text{TP}(\tau) + \left(\frac{1}{\tau} \int_0^\tau [1, \exp(-\phi \Delta t)] d\tau \right) \beta(t) - [1, 1] \frac{1}{\tau} \left(\int_0^\tau \exp(-\kappa \tau) d\tau \right) \beta(t) \\
&= \text{TP}(\tau) + \left[1, \frac{1}{\tau} F(\phi, \tau) \right] \beta(t) - [1, 1] \left(\frac{1}{\tau} \left\{ -[\kappa]^{-1} \exp(-\kappa \tau) \Big|_0^\tau \right\} \right) \beta(t) \\
&= \text{TP}(\tau) + \left[1, \frac{1}{\tau} F(\phi, \tau) \right] \beta(t) - [1, 1] [\kappa \tau]^{-1} [I - \exp(-\kappa \tau)] \beta(t) \\
&= \text{TP}(\tau) + \left(\left[1, \frac{1}{\tau} F(\phi, \tau) \right] - [1, 1] [\kappa \tau]^{-1} [I - \exp(-\kappa \tau)] \right) \beta(t)
\end{aligned}$$