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Optimal Grid Selection in Numerical Approaches to Dynamic Heterogeneous Agent Macroeconomic Models*

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Abstract

This paper unites numerical literature concerning the optimal linear approximation of convex functions with theory on the consumption savings problems of households in macro economies with idiosyncratic risk and incomplete markets. Construction of a grid for the optimal linear approximation of household savings behavior is characterized in an environment with income fluctuations and a single savings asset. For wealthy households, the grid is characterized asymptotically as having a density which decreases in household wealth. For domains which include resource poor households, the optimal grid is approximated numerically and is seen to have non-monotonic grid point density for standard parameters.

This result contradicts conventional rules for constructing grids for discretized state spaces in basic heterogeneous agent models, namely the practice of constructing grids with density which increases monotonically near the borrowing constraint. Rather, the optimal grid has high density where the curvature of the household's savings function is largest. Meanwhile, curvature increases with wealth for the very poor provided the risky income asset is of sufficiently high quality. Quantitatively, the approximate optimal grid is seen

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to outperform standard grid constructs according to a variety of accuracy measures, in terms of both closeness of approximation and satisfaction of the household optimality condition. This accuracy improvement comes at the cost of increased computational time, and efficiency improving alternatives for automating grid choice are given.

Non-technical summary

A well-known fact about household savings behaviour is that it depends on an individual's own wealth holdings and income expectations. In understanding how these decisions aggregate and respond to policy, we must therefore confront the substantial heterogeneity in these features which is present in the macro economy. One persistent challenge in the literature which aims to provide this understanding is the need to keep track of and forecast the decisions of every single individual in the economy. Such a feat is not possible in practice, so that some grouping of individuals is typically needed to sidestep the issue.

This article considers the dividing of households into representative groups to optimally represent savings decisions across the wealth distribution. Such a division is guided by the principle that many groups are needed where there is a large amount of variation in savings propensities. Combining numerical analysis with theory on consumption behaviour, it is shown in the context of a simple model that the savings behaviour of wealthy and poor households can be captured by including a single representative for each category. Meanwhile, several groups are needed to describe the behaviour of the middle class. Middle class households face uncertain future income comparable in size to current wealth holdings, leading to a wide range of savings propensities when compared to poor hand-to-mouth individuals and wealthy savers.

1 Introduction

Robust and accurate numerical methods are indispensable in the solving of heterogeneous agent macroeconomics models. This class of models features substantial nonlinearity and high dimensionality inherent in the optimization problems facing the various economic agents, which are largely absent in the representative agent framework. Consequently, the numerical methods applied must be finely tuned to each individual model and its parameters. This has led to a large body of literature in recent decades aiming to address the accuracy, speed, and versatility of the numerical solution of baseline heterogeneous agent models [Krusell & Smith (1998), Maliar et al. (2010), Reiter (2009), Young (2010), Den Haan (2010*a*)]. A recurring challenge to applying these solution methods is that their implementation can be ad-hoc in places where the researcher is lacking theoretical guidance on how to proceed optimally, with potential consequences for the accuracy and efficiency of the method being used.

An example of the sort of discretionary choice one faces in the numerical approaches to heterogeneous agent models is in constructing a discrete approximation to an economic agent's continuous state space. While some rules of thumb do apply to this selection, for example that the selected grid should include the deterministic steady state of the economy, there is typically not a systematic way to pin down the breadth of the discrete space needed nor the distribution of the discrete states within the chosen domain. While solution accuracy can be ensured by trial and error (see, for example, Maliar et al. (2010)), this procedure is not particularly theoretically motivated or efficient, and moreover cannot produce broadly applicable results. One example of an algorithm which automates the selection of a grid for interpolation is the endogenous grid of Carroll (2006). This method greatly increases the speed and simplicity of the solution, selecting grid points in such a way as to bypass root finding in a household's condition of optimality. Nonetheless, the method involves interpolating back to a fixed grid at each iteration step, and this grid must be specified in advance.

This article aims to connect literature in numerical analysis which addresses optimal approximation of continuous functions with incomplete markets heterogeneous agent macro economic models. In particular, we consider methods of Gavrilović (1975) for the approximation of a known function by way of a piecewise linear function, which is optimal in the sense of minimizing the worst absolute error. Specifically, we consider the theoretical and numerical results of Gavrilović (1975) in the context of the savings decision of a finitely lived household facing incomplete markets along with a simple form of income fluctuations of the type considered in Huggett (1993) and Aiyagari (1994).

Conventional wisdom has long held that wealthy households in models of this type behave linearly, while all nonlinearity in savings behavior occurs at low wealth levels. The convention then dictates that the implementation of a given solution to such a model should place many grid points near the constraint on household borrowing, where the curvature is the largest. By combining the conditions of optimality of Gavrilović (1975) with the theory of Chipeniuk et al. (2016), we establish a clear asymptotic story for the optimal linear approximation of the savings behavior of wealthy households in our simplified model. True to the conventional wisdom, we find that in the appropriate asymptotic sense the density of the optimal grid is decreasing in household wealth when the households of interest are sufficiently wealthy, regardless of the model parameters. In particular, the asymptotic features of the optimal grid are independent of the qualities of the risky income process.

We then implement a modified version of the numerical approach of Gavrilović (1975) to investigate the properties of an approximate optimal grid which includes poor and middle class households in the model. For these agents we find that the conventional approach to grid selection may fail to be optimal in a broad qualitative sense: the optimal grid density need not be decreasing in household wealth. While the algorithm does indeed place many grid points in sections of the choice set where savings displays high curvature, the curvature itself is nondecreasing in wealth for poor agents. A theoretical investigation of the analogous static model reveals this non-monotonicity to be tied to the quality of risky household income: if this asset is of low quality, as characterized by a small Sharpe ratio, it does not factor heavily into the decisions of any household, so that all behave as relatively wealthy agents. In the case of a high quality asset, the poor make very different decisions from the wealthy, instead acting approximately as a representative hand-to-mouth individual, and the transition region - the middle class - faces high curvature in their decisions.

Quantitatively, we compare the accuracy of the approximation obtained using the Gavrilović grid relative to that obtained using standard Linear, Logarithmic, and Polynomial grid constructions. As expected, the former construction is seen to outperform the others along its target measure of minimizing the worst absolute error, but also along several alternative measures of accuracy, both in terms of nearness of approximation and the error introduced into the household Euler equation, which is the key condition of optimality governing the consumption savings decision. These results suggest a role for considering such a construction in more complicated applied frameworks than the simple model presented here, however the gains in accuracy come with a significant increase in computational time. We investigate various mechanisms for reducing this loss of efficiency. By specifying the desired approximation accuracy, rather than the number of grid points, it is possible to replace Gavrilović (1975)'s iterative method with a faster direct algorithm. Moreover, it can be argued that this modification of the grid selection problem is more in line with the practical needs of a modeler.

In the next section, we begin by summarizing the content of Gavrilović (1975), after which we introduce the income fluctuation problem which forms our example case. We then conduct the asymptotic analysis of the optimal grid for wealthy households, and examine the curvature of the savings function for poor and middle class households, analytically in a static model and numerically in dynamic models. We then conclude by constructing the optimal grid for equilibrium values of aggregates and evaluating it's characteristics and accuracy.

2 Background on Optimal Grids for Linear Approximations

We begin by briefly describing the method of Gavrilović (1975) for constructing an optimal grid for the linear approximation of a known twice continuously differentiable, strict convex or concave function $f : \mathbb{R} \to \mathbb{R}$ on a closed interval $[a, b]$.[1] Specifically, we are interested in approximating f by the piecewise linear function ℓ given by

$$\ell(x) = y_n + \frac{y_{n+1} - y_n}{x_{n+1} - x_n}(x - x_n) \quad \text{if } x \in [x_n, x_{n+1}] \text{ for } n = 0, ..., N$$

[1] The smoothness and strictness assumptions may be relaxed in the analysis up to (2)-(4) below.

in which we are free to choose $x_1, ..., x_N$ as well as $y_0, ..., y_{N+1}$ while fixing $x_0 = a$ and $x_{N+1} = b$.[2] We are interested in choosing these grid points such that the approximation is optimal in the sense that it minimizes the largest pointwise absolute difference between f and ℓ. That is, we are solving the problem

$$\min_{\substack{x_1,...,x_N \\ y_0,...,y_{N+1}}} \max_{x \in [x_0, x_{N+1}]} |f(x) - \ell(x)|$$

Let this optimal value be denoted $V_N(x_0, x_N)$ and further denote

$$\ell_n(x) = y_n + \frac{y_{n+1} - y_n}{x_{n+1} - x_n}(x - x_n) \quad \text{if } x \in [x_n, x_{n+1}]$$
$$V_0(x_n, x_{n+1}) = \min_{y_n, y_{n+1}} \max_{x \in [x_n, x_{n+1}]} |f(x) - \ell_n(x)|$$

Then V_0 gives the error for an optimal selection of y_n and y_{n+1} once the endpoints x_n and x_{n+1} have been fixed.

With this notation in hand, the main theorem of Gavrilović (1975) states that the solution to the optimal grid choice problem will have

$$V_N(x_0, x_{N+1}) = V_0(x_n, x_{n+1}) \quad \text{for every } n = 0, ..., N \tag{1}$$

and moreover that this worst value will be encountered in three places in each subinterval: the two endpoints x_n and x_{n+1} as well as an intermediate point x^* (see Figure 1).

The second statement above can be expressed as three optimality conditions as follows:

$$V_0(x_n, x_{n+1}) = |f(x_n) - y_n| \tag{2}$$
$$V_0(x_n, x_{n+1}) = |f(x_{n+1}) - y_{n+1}| \tag{3}$$
$$V_0(x_n, x_{n+1}) = \max_{x \in (x_n, x_{n+1})} |f(x) - \ell_n(x)| \tag{4}$$

[2] It is common in economic applications to interpolate the function in question, in which case $y_n = f(x_n)$. It is therefore natural to ask how interpolation, which essentially imposes an additional constraint on the problem, changes the optimal approximation. However, following the induction in Gavrilović (1975), one can see that interpolation would simply shift the optimal approximation up by the worst error while keeping the endpoints $x_0, ..., x_{N+1}$ in the same place, thereby doubling the error while requiring the same amount of work to find. This is evident in the base case $N = 0$, and amounts to replacing (2) and (3) below by the constraints which fix the endpoints of each piecewise linear segment to the function values.

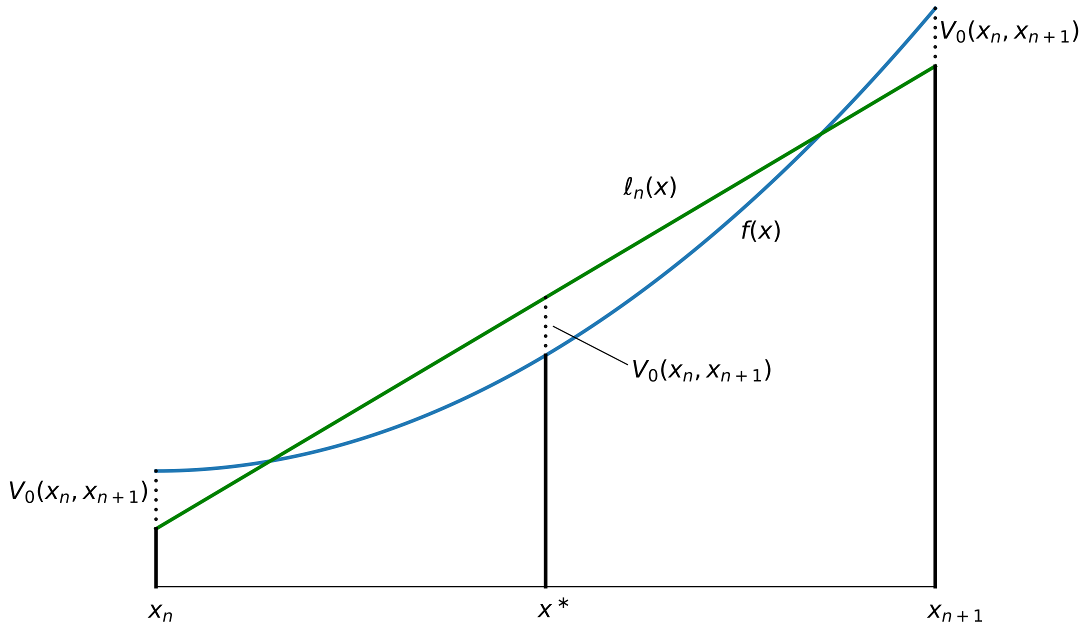

Figure 1: Reproduction of Figure 1 in Gavrilović (1975) showing the optimal linear approximation of a convex function on a single subinterval.

Along with the knowledge that $V_0(x_n, x_{n+1})$ identical for every value of n, these conditions can be used to pin down the optimal values of $x_1, ..., x_N$ and $y_0, ..., y_{N+1}$. Specifically, letting x^* denote the location of the critical point for the maximization in (4), the first order condition reads

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = f'(x^*) \quad \Longrightarrow \quad x^* = (f')^{-1}\left(\frac{y_{n+1} - y_n}{x_{n+1} - x_n}\right) \tag{5}$$

where f' is invertible as a consequence of the strict convexity or concavity of f. Then, for a convex function we get

$$\begin{aligned} V_0(x_n, x_{n+1}) &= \ell_n(x^*) - f(x^*) \\ &= y_n + \frac{y_{n+1} - y_n}{x_{n+1} - x_n}(x^* - x_n) - f(x^*) \\ &= f(x_n) - V_0(x_n, x_{n+1}) + \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}(x^* - x_n) - f(x^*) \end{aligned}$$

where the last equality used the condition that the maximal error is encountered at the endpoints. Rearranging slightly,

$$2V_0(x_n, x_{n+1}) = f(x_n) + \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}(x^* - x_n) - f(x^*) \tag{6}$$

Recalling (5) and using it above to rewrite x^* as

$$x^* = (f')^{-1}\left(\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}\right) \tag{7}$$

we can obtain an expression for $V_0(x_n, x_{n+1})$ entirely in terms of its arguments x_n and x_{n+1}, as well as the known function f.

Recalling that the optimal choice of $x_1, ..., x_N$ requires all the values $V_0(x_n, x_{n+1})$ to be the same (say V_0) will allow us to find $x_1, ..., x_N$ if f is sufficiently simple. For example, certain functions result in expressions for $V_0(x_n, x_{n+1})$ which telescope when summed over n, giving in turn an expression for V_0 entirely in terms of the exogenous endpoints x_0 and x_{N+1} and the number of grid points N. It is then possible to solve

$$V_0(x_0, x_1) = V_0 \tag{8}$$

for x_1, after which we can use this value to solve $V_0(x_1, x_2) = V_0$ for x_2, and so on. Once the endpoints have been found, the approximation values $y_0, ..., y_{N+1}$ are easily backed out from (2) and (3).

We now give an explicit analytical example of this procedure which will prove to be relevant to the discussion of household optimization problems in heterogeneous agent macroeconomic models.

Example 1: f(x)=1/x. With $f(x) = 1/x,\ x > 0$, we have $f'(x) = -1/x^2,\ x > 0$, so we can solve for x^* from

$$-\frac{1}{(x^*)^2} = \frac{\frac{1}{x_{n+1}} - \frac{1}{x_n}}{x_{n+1} - x_n} \tag{9}$$

giving

$$x^* = \left(\frac{x_{n+1} - x_n}{\frac{1}{x_n} - \frac{1}{x_{n+1}}}\right)^{1/2} \tag{10}$$

Substituting into (6),

$$2V_0(x_n, x_{n+1}) = \frac{1}{x_n} + \frac{\frac{1}{x_{n+1}} - \frac{1}{x_n}}{x_{n+1} - x_n}\left(\left(\frac{x_{n+1} - x_n}{\frac{1}{x_n} - \frac{1}{x_{n+1}}}\right)^{1/2} - x_n\right) - \left(\frac{\frac{1}{x_n} - \frac{1}{x_{n+1}}}{x_{n+1} - x_n}\right)^{1/2}$$

Simplifying, and writing the outcome as a single fraction we get

$$
\begin{aligned}
2V_0(x_n, x_{n+1}) &= \frac{1}{x_n} - \frac{\frac{1}{x_{n+1}} - \frac{1}{x_n}}{x_{n+1} - x_n} x_n \\
&= \frac{\frac{1}{x_n}(x_{n+1} - x_n) - x_n \left(\frac{1}{x_n} - \frac{1}{x_{n+1}} \right)}{x_{n+1} - x_n} \\
&= \frac{\frac{x_{n+1}}{x_n} - 2 + \frac{x_n}{x_{n+1}}}{x_{n+1} - x_n} \\
&= \frac{x_{n+1}^2 - 2x_n x_{n+1} + x_n^2}{x_n x_{n+1}(x_{n+1} - x_n)}
\end{aligned}
$$

Factoring the numerator, we therefore obtain

$$2V_0(x_n, x_{n+1}) = \frac{x_{n+1} - x_n}{x_n x_{n+1}}$$

Hence we have

$$V_0(x_n, x_{n+1}) = \frac{1}{2} \left(\frac{1}{x_n} - \frac{1}{x_{n+1}} \right)$$

Applying the optimality condition that each of these is equal to V_0 and summing over n, we get

$$(N+1)V_0 = \frac{1}{2} \left(\frac{1}{x_0} - \frac{1}{x_{N+1}} \right) \tag{11}$$

Hence x_1 will be given by solving

$$\frac{1}{2} \left(\frac{1}{x_0} - \frac{1}{x_1} \right) = \frac{1}{2(N+1)} \left(\frac{1}{x_0} - \frac{1}{x_{N+1}} \right) \tag{12}$$

which results in

$$x_1 = \frac{1}{\left(\frac{N}{N+1} \right) \frac{1}{x_0} + \left(\frac{1}{N+1} \right) \frac{1}{x_{N+1}}} \tag{13}$$

Now that we have x_1, we can find x_2 by solving

$$\frac{1}{2} \left(\frac{1}{x_1} - \frac{1}{x_2} \right) = \frac{1}{2(N+1)} \left(\frac{1}{x_0} - \frac{1}{x_{N+1}} \right) \tag{14}$$

which readily gives

$$x_2 = \frac{1}{\left(\frac{N-1}{N+1} \right) \frac{1}{x_0} + \left(\frac{2}{N+1} \right) \frac{1}{x_{N+1}}} \tag{15}$$

Continuing this procedure we obtain

$$x_j = \frac{1}{\left(\frac{N+1-j}{N+1}\right)\frac{1}{x_0} + \left(\frac{j}{N+1}\right)\frac{1}{x_{N+1}}}, \quad j = 0, 1, ..., N+1 \tag{16}$$

Thinking of this as a function of a continuous variable j, we can differentiate twice to get

$$\frac{dx_j}{dj} = \frac{x_j^2}{N+1}\left(\frac{1}{x_0} - \frac{1}{x_{N+1}}\right)$$
$$\frac{d^2x_j}{dj^2} = \frac{2x_j^3}{(N+1)^2}\left(\frac{1}{x_0} - \frac{1}{x_{N+1}}\right)^2 > 0$$

Since the second derivative here is positive, we can conclude that optimal grid points are spaced farther apart for larger values of j. □

In more sophisticated examples, solving the conditions (1)-(4) analytically proves infeasible, in which instance Gavrilović (1975) proposes the following numerical procedure:

Algorithm 1: Optimal Grids for Linear Approximations.

1) Guess a value for $V_0 = V_0(x_n, x_{n+1})$.
2) Starting with $n = 0$ and taking V_0 as given, solve (2)-(4) for x_{n+1} and y_{n+1} for each $n = 0, ..., N$. More precisely, one can solve the reduced conditions (7) and (6) for x_{n+1} given x_n, and y_{n+1} can be determined from (4).
3) Check whether the resulting value for x_{N+1} is, to within tolerance, equal to b. If so, stop. If not, adjust the guess for V_0, and return to 2.

Implementing this algorithm requires choosing a method for solving the equations in step 2, as well as choosing a method to update the guess for V_0, such that the procedure will converge. Both of these amount to one dimensional root finding problems with unique solutions, which can be solved efficiently. Gavrilović (1975) proposes using the Newton-Raphson method in both instances. In the case of updating the guess for the worst approximation error, this involves computing the derivative of x_{N+1} with respect to V_0, which can be achieved by taking the implicit derivatives with respect to V_0 in (7) and (6). The resulting system can be solved for $\partial x_{n+1}/\partial V_0$ in terms of $\partial x_n/\partial V_0$, so that $\partial x_{N+1}/\partial V_0$ is determined in the last step of the iteration.

An alternative approach, and the one we apply below, is to exploit the one dimensional nature of the problem and apply binary search to solve the optimality conditions and update the approximation error. In addition to not needing as input the derivatives with respect to the worst approximation error, this method has the advantage that it is guaranteed to converge with a continuous problem, provided we can find an interval which contains the solution. It is also extremely efficient, as it eliminates half of the search space in each step.

In the context of macroeconomic modeling, the key weakness of the discussion so far, and in particular that of Algorithm 1, is that it requires us to be able to compute values of the function f. When solving an economic model with a given numerical approach, it is typical that interpolation is used as part of an iterative procedure to approximate an object of interest, such as a savings, consumption, or value function, and that the exact analytic form of this object is unknown prior to constructing the approximation. As a result, we cannot directly apply Algorithm 1 to problems such as the one that we outline in the next section while simultaneously avoiding the curse of dimensionality.

We sidestep this issue using two separate approaches. First, although we do not have access to a closed form of the savings function of a household in our model, we can combine the analytic optimality conditions for the linear approximation of Gavrilović (1975) with known properties of this function to theoretically investigate the shape of the optimal grid without resorting to numerics at all. This allows us to characterize the optimal approximation for wealthy households. Second, in Section 5.1 we examine modifications of Algorithm 1 which may be run alongside the iterative solution to an economic model. Unfortunately, these methods prove to be computationally inefficient, limiting their practical value. On the other hand, they do allow us to investigate the shape and accuracy gains from the optimal grid, and thereby shed some light on why certain choices of ad-hoc grids may be expected to perform better than others. A surprising outcome is that the optimized grids for linear approximation should not necessarily place most of the grid points near the borrowing constraint, even in an extremely simplified model. We next turn to outlining the structure of such a model.

3 A Simple Incomplete Markets Life Cycle Model

In our consideration of optimal grid selection in heterogeneous agent macroeconomic models, we will consider a simple version of the problem of a finitely lived household facing incomplete markets and idiosyncratic uncertainty. In particular, time is discrete and finite, consisting of T periods indexed by $t = 1, ..., T$. A given household will encounter fluctuations in a period by period endowment while smoothing its consumption path through the use of a single, non state contingent savings vehicle which may be held in any finite quantity, up to some borrowing limit.

A standard approach in numerically solving such a problem is to discretize the space of possible asset holdings by specifying a finite set of current resource holdings at which the researcher pins down household savings behavior while interpolating this behavior at intermediate points. This raises the question of how to distribute the grid points in order to reduce numerical error, for example by making use of the scheme presented in the previous section.

Turning to a specific formulation, we will consider a household valuing consumption according to

$$U(c_1, c_2, ..., c_T) = \mathbb{E}_1 \sum_{t=1}^{T} \beta^{t-1} u(c_t) \tag{17}$$

where $0 < \beta < 1$ is the intertemporal discount factor and period utility is logarithmic, $u(c_t) = \log(c_t)$.

Household income in each period is composed of proceeds from a single savings asset and an endowment which is driven by an individual and exogenous stochastic process. We denote the level of savings brought into period $t+1$ by k_t and the endowment in period t by ℓ_t. Since our current interest is in the broad features of optimally constructed asset grids, we will fix the endowment process as taking on one of two possibilities in each period independent of previous outcomes,

$$\ell_t = \begin{cases} \ell_{\text{high}} & \text{with probability } p \\ \ell_{\text{low}} & \text{with probability } 1-p \end{cases}$$

Let R_t denote the value of savings relative to consumption in period t, and suppose that the savings asset depreciates at the rate δ. Then denoting by

W_t the price of a unit of endowment in terms of the consumption good, the period resource constraint can be written as

$$c_t + k_t \leq (1 - \delta + R_t)k_{t-1} + W_t \ell_t \tag{18}$$

where the left hand side is expenditures at time t and the terms on the right hand side are savings (net of depreciation) and endowment income, respectively. We will assume that households are constrained in their borrowing up to a natural borrowing limit, $\underline{k}_t$, which in the parameterization below will be equal to 0.

Absent aggregate uncertainty we may reformulate the household's sequential problem of maximizing (17) with respect to (18) as a collection of dynamic programming problems for value functions $V^{(t)}(x_t) := V^{(t)}(x_t, R_t, W_t, ..., R_T, W_T)$ as

$$\begin{aligned} V^{(t)}(x_t) \;&=\; \max_{c_t, k_t} \left(u(c_t) + \beta \mathbb{E}_t V^{(t+1)}\left(x_{t+1}\right) \right) \qquad (19) \\ \text{subject to} \quad & c_t + k_t \leq x_t \\ & x_{t+1} = (1 - \delta + R_{t+1})k_t + W_{t+1}\ell_{t+1} \\ & k_t \geq \underline{k}_t \end{aligned}$$

for $t = 1, ..., T$, along with the terminal condition $V^{(T+1)} \equiv 0$. Owing to our choice of utility function, borrowing constraint, and finite horizon, one can show by induction that there is a unique solution to this sequence of problems.[3] This solution indicates a sequence of savings functions $k^{(t)}(x_t, R_t, W_t, ..., R_T, W_T)$, where below we will suppress the dependence on the price sequence for simplicity of notation. From our assumption of a natural borrowing limit along with the functional form of the utility function, a necessary condition for optimality comes in the form of the intertemporal Euler equations

$$\frac{1}{x_t - k^{(t)}(x_t)} = \beta \mathbb{E}_t \left(\frac{1}{k^{(t)}(x_t) + w_{t+1} - \frac{k^{(t+1)}(x_{t+1})}{1-\delta+R_{t+1}}} \right) \tag{20}$$

where $w_{t+1} = W_{t+1}\ell_{t+1}/(1 - \delta + R_{t+1})$ and $k^{(T+1)} \equiv 0$.

With $T = 2$, we can solve the single resulting Euler equation explicitly for its nonnegative root using the quadratic formula, giving the hyperbolic savings function

$$k^{(1)}(x_1) = \frac{Ax_1 - B - C + \sqrt{(Ax_1 - B + C)^2 - \mathrm{Var}(n_2)}}{2} \tag{21}$$

[3] See Chipeniuk et al. (2016), Proposition 1.

where

$$\begin{aligned}
A &= \frac{\beta}{1+\beta} \\
B &= \frac{1}{1+\beta}\mathbb{E}_1(n_2) \\
C &= w_{\text{low}} + \frac{\text{Var}(n_2)}{\mathbb{E}_1(n_2)} \\
n_2 &= w_2 - w_{\text{low}} \\
w_{\text{low}} &= \frac{W_2}{(1-\delta+R_2)}\ell_{\text{low}}
\end{aligned}$$

In the dynamic case, Chipeniuk et al. (2016) establishes under general conditions that the savings function can be written as an explicit linear asymptote plus an implicit convex nonlinear term $\epsilon^{(t)}$ which is strictly decreasing to zero as wealth increases. Writing out the functional form of the asymptote for the case of logarithmic period utility considered here, we have

$$\begin{aligned}
k^{(t)}(x_t) = & \frac{\beta + ... + \beta^{T-t}}{1+\beta+...+\beta^{T-t}}x_t \\
& - \frac{1}{1+\beta+...+\beta^{T-t}}\mathbb{E}_t\left(\sum_{s=t+1}^{T}\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) \\
& + \epsilon^{(t)}(x_t)
\end{aligned} \tag{22}$$

In the above expansion the leading term on the right may be interpreted as permanent savings out of current resources, while the second term represents a downward adjustment which is driven by the incentive to borrow against future expected discounted income. A summary of the steps used to derive this expression in the current setting is given in Appendix A.

It follows from the convexity of $\epsilon^{(t)}$ that we may, in principle, apply the results of Gavrilović (1975) to this function.[4] Interpreted in this context, the optimality conditions for the selection of a linear approximation tell us how to select an optimal representative household for a wealth bin $[x_n, x_{n+1}]$. This optimal representative for the bin has a savings function with the same propensity to save as the average across the bin (recall Figure 1). There

[4] One could equivalently apply the concave version of Gavrilović (1975) to the consumption function, since consumption and savings in this model satisfy a linear relationship in the resource constraint. An alternative approach in this model, which we do not pursue here, would be to apply the method to the concave value function in a value function iteration.

are two extreme cases, namely the household in the bin which has a savings function with the exact same slope as this, and the two endpoints where the slopes differ most dramatically from the representative. The optimal representative in the sense we are discussing here exactly balances the loss from these extremes.

We can close the economy as it has been given thus far through a variety of market structures. In order to fix ideas, in the numerical exercises we will follow Aiyagari (1994) and interpret savings as capital and the endowment as labor, which are both rented to perfectly competitive firms who use Cobb-Douglas technology to produce output every period. Formulating the stationary equilibrium then requires us to find prices R and W which clear capital, labor, and output markets under optimizing behavior by firms and households as well as a stationary distribution of households across ages, capital, and endowments which is moreover consistent with the household decisions. These equilibrium objects will be contingent on the distribution of households across capital and endowments at birth, which we take to be point masses of employed and unemployed households with no capital, the weights of these points being equal to the employment and unemployment rates, respectively.

The numerical model is calibrated to closely match an annualized, finite horizon version of other numerical studies of baseline models, for example Maliar et al. (2010) and Den Haan (2010*b*). Specifically, we take $T = 60$ for the time horizon[5], $\beta = 0.96$ for the household discount factor, $\alpha = 0.36$ for the Cobb-Douglas weight on capital, and $\delta = 0.1$ for the depreciation rate of capital. We interpret the bad endowment state as unemployment, so that $\ell_{\text{low}} = 0$. We set an unemployment rate of 0.04, so that $p = 0.96$, and choose the good endowment in order to normalize total labor efficiency to 1, $\ell_{\text{high}} = 1/0.96 \approx 1.04$. In some exercises we also consider a high quality endowment which is realized with higher probability, $p = 0.999$.

The equilibrium was computed in the computing language Julia using typical Euler equation iteration and density propagation methods along with a Walrassian auctioneer mechanism for market clearing. In the baseline parameterization, the auctioneer mechanism clears the capital and labor markets to within relative errors of 10^{-6}, at which point the goods market for firm output clears via Walras' law to within a relative error of 10^{-13}. The levels

[5] The numerical results presented apply equally to the savings function in a model with infinitely lived households. In particular, as will be seen in the figures below, increasing the horizon tends to exacerbate the counterintuitive properties of the optimal linear approximation of household savings.

of aggregate capital and consumption at the equilibrium prices are approximately 3.78 and 1.24 respectively. With the higher probability of employment, equilibrium aggregate capital and consumption are approximately 3.89 and 1.28 respectively.

4 Optimal Linear Approximations and Curvature of Household Savings

In this section, we derive some properties and intuition regarding the shape of an optimal grid for resources x_t on which to linearly approximate $k^{(t)}$ for the household's problem discussed in the previous section.

We begin by noting that if we express a function as a linear component plus a nonlinear error, we need only consider the nonlinear error when selecting grid points. Using this property, we are then able to establish the asymptotic shape of the optimal grid in the limit where we consider wealthy individuals. The asymptotic result confirms the conventional wisdom that grid points should become increasingly farther apart as household resources increase, at least in this limit.

4.1 Optimal Grid Selection with a Linear Function Plus a Nonlinear Error

We now consider the problem of finding the optimal grid to approximate the function $k : \mathbb{R} \to \mathbb{R}$ which we write in the generic linear-plus-error form

$$k(x) = Ax - B + \epsilon(x) \tag{23}$$

where A and B are constants independent of x. Here we interpret ϵ as some nonlinear function, where in the savings function described in the previous section we have $\epsilon(x) \to 0$ as $x \to \infty$. Recall (6) and (7), which here we write as

$$2V_0(x_n, x_{n+1}) = k(x_n) + \frac{k(x_{n+1}) - k(x_n)}{x_{n+1} - x_n}(x^* - x_n) - k(x^*)$$

and

$$\frac{\partial k}{\partial x}(x^*) = \left(\frac{k(x_{n+1}) - k(x_n)}{x_{n+1} - x_n}\right)$$

respectively. In the first of these, we write k in linear-plus-error form to get

$$\begin{aligned}2V_0(x_n, x_{n+1}) =& [Ax_n - B + \epsilon(x_n)] + A(x^* - x_n) + \frac{\epsilon(x_{n+1}) - \epsilon(x_n)}{x_{n+1} - x_n}(x^* - x_n)\\ &- [Ax^* - B + \epsilon(x^*)]\\ &= \epsilon(x_n) + \frac{\epsilon(x_{n+1}) - \epsilon(x_n)}{x_{n+1} - x_n}(x^* - x_n) - \epsilon(x^*)\end{aligned}$$

In the second we do the same, getting

$$A + \frac{\partial \epsilon}{\partial x}\epsilon(x^*) = \frac{A(x_{n+1} - x_n) + \epsilon(x_{n+1}) - \epsilon(x_n)}{x_{n+1} - x_n}$$

or

$$\frac{\partial \epsilon}{\partial x}(x^*) = \frac{\epsilon(x_{n+1}) - \epsilon(x_n)}{x_{n+1} - x_n}$$

Hence the optimal grid points $x_1, ..., x_N$ satisfy the conditions

$$2V_0(x_n, x_{n+1}) = \epsilon(x_n) + \frac{\epsilon(x_{n+1}) - \epsilon(x_n)}{x_{n+1} - x_n}(x^* - x_n) - \epsilon(x^*) \tag{24}$$

$$\frac{\partial \epsilon}{\partial x}(x^*) = \frac{\epsilon(x_{n+1}) - \epsilon(x_n)}{x_{n+1} - x_n} \tag{25}$$

along with the usual condition that $V_0(x_n, x_{n+1}) \equiv V_0$. These are just the conditions used to pin down the optimal grid for the nonlinear error, so it suffices to restrict our attention to this error. Summarizing, we have the following.

Lemma 1: Optimal Grids With Nonlinear Errors. For a function k of the form (23), the optimal grid for linearly approximating k (in the sense of Gavrilović (1975)) is the same as that for approximating the nonlinear component ϵ.

It therefore suffices to restrict our attention to the nonlinear term for the simple model described in the previous section.

4.2 Optimal Grids For Wealthy Households

We now consider the selection of an optimal discretized resource space for linearly approximating the savings function $k^{(t)}(x_t)$ for households whose wealth is sufficiently high.

While the optimality conditions spelled out in Gavrilović (1975) can be solved explicitly for simple cases like that in Example 1, this is not possible for more complicated examples such as the savings function expressed in (21). Additionally, obtaining analytic results using these conditions is made difficult by the fact that closed form solutions for $k^{(t)}$ are inaccessible.

However, in the limit of an extremely wealthy household the savings behavior becomes approximately linear and does so in a monotonic way, raising the possibility for characterizing the grid in this limit. In fact, for wealthy households, the nonlinear error in savings behavior is approximated well by the function of Example 1, and consequently the optimal grid in this region is as well. The precise statement is the following.

Theorem 1: Optimal Grids For Wealthy Household. The optimal grid points x_j for a linear approximation (in the sense of Gavrilović (1975)) of the savings functions $k^{(t)}$ which solve the household's dynamic programming problems (19) satisfy

$$x_j = \frac{1 + O(1/x_0)}{\left(\frac{N+1-j}{N+1}\right)\frac{1}{x_0} + \left(\frac{j}{N+1}\right)\frac{1}{x_{N+1}}}$$

Proof. See Appendix A. □

To gain some intuition for the proof of this result, consider the case $T = 2$. For notational simplicity we will omit time subscripts. Specifically, let $x := x_1$ denote initial resources in this setting, and let $\epsilon(x) := \epsilon^{(1)}(x)$ denote the nonlinear term in the corresponding savings function $k(x) := k^{(1)}(x)$. In this case, it can be shown that the error can be written implicitly in terms of the savings function itself,

$$\epsilon(x) = \left(\frac{1}{1+\beta}\right)\frac{\mathrm{Var}_1(n_2)}{k(x) + w_{\mathrm{low}} + \frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)}}$$

and, once more,

$$n_2 = w_2 - w_{\mathrm{low}}. \tag{26}$$

Writing

$$\epsilon(x) = \left(\frac{1}{x}\right)\frac{\mathrm{Var}_1(n_2)}{\frac{k(x)}{x} + \frac{w_{\mathrm{low}} + \frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)}}{x}}$$

and recalling that $k(x)$ is approximately linear as $x \to \infty$, we see that $k(x)/x$ must approach some constant in this limit while the second term in the denominator vanishes. In other words, as $x \to \infty$ there is a constant C such that

$$\epsilon(x) \sim C/x \tag{27}$$

Likewise, we can observe that

$$\frac{\partial \epsilon}{\partial x} \sim -C/x^2$$

From Lemma 1 we know that the optimal grid for k is simply that for ϵ, so that the choice of grid depends (from the corresponding optimality conditions) only on ϵ and ϵ', which in turn we have now seen behave in the limit as if $\epsilon(x)$ were equal to C/x.

While we only give the asymptotic result for a very stylized environment, it likely extends to much richer settings. The expansion (23) can be derived with far weaker restrictions on the stochastic process for labor income. Moreover, this same expression holds when one adds aggregate uncertainty which impacts household expectations about future prices. The only key restrictions on idiosyncratic and aggregate stochastic processes that is needed is that they have a finite number of positive probability outcomes in each period. Still more, an analogous expansion holds with general CRRA period utility functions. In all of these cases, the nonlinear component exhibits geometric decay as in (27), and the above result will apply to a grid constructed for the household's individual resource space.

In addition, while we have assumed a natural borrowing limit to ensure that the intertemporal Euler equation holds across the entire household resource space, in the case of a constraint which binds for some feasible allocation of resources the Euler equation will still hold away from the binding region. In particular, the algebraic manipulations used to prove the asymptotic result will still go through, insofar as the borrowing constraint fails to bind for the wealthy. Thus the result is seemingly robust to many straightforward generalizations of baseline heterogeneous agent models. On the other hand, it is not immediately clear how to extend the derivation, nor even the statement of the result to cases in which the modeler is concerned with constructing a discrete representation of a multidimensional state space.

As mentioned previously, the theorem of this section essentially confirms the convention of grid points which become more spread out as one sees in, for example, logarithmic or polynomial grids. Following the standard intuition, the wealthier households are, the less they are concerned with risky income and the more closely they behave to a single representative, permanent income agent. Consequently, we need fewer distinct representative bins as we extend into the right tail of the wealth distribution. We now give analytic and numerical evidence to suggest that, when considering less wealthy households, this convention can break down.

4.3 Curvature of Household Savings

The next section complements the theoretical results for rich households given above with a numerical investigation of optimal grid properties for approximating the savings behavior of households across the entire distribution of wealth in the model of Section 3. Prior to giving the results of these calculations, however, we provide intuition and evidence for why an optimal linear approximation for household savings in a basic model may cluster grid points away from the borrowing constraint.

We establish, rigorously in the two period case and numerically for longer horizons, that the curvature of the savings function need not be monotonic in household resources. Given the intuition that optimal linear approximations should have high resolution where curvature is high, as encompassed by Lemma 1, we should not expect optimal grids to display decreasing grid point density for less wealthy households. Indeed, this will be one of the key observations in the qualitative exercises which follow.

The result in the static setting is the following.

Theorem 2: Non-Monotonic Second Derivative With Two Periods. The savings function $k^{(1)}$ which solves the household's dynamic programming problem (19) with $T = 2$ has a non-monotonic second derivative if and only if

$$\frac{\mathbb{E}_1(n_2)^2}{\mathrm{Var}_1(n_2)} > 1 + \beta \tag{28}$$

Proof. See Appendix A. □

Note that, as a Corollary of this theorem, with an appropriate adjustment of subscripts the same is true for the final savings decision $k^{(T-1)}$ of a typical household in the general life cycle model described in the previous section.

The proof of Theorem 2 is a straightforward application of the Intermediate Value Theorem to the third derivative of k, which is just equal to the third derivative of the nonlinear portion ϵ. It turns out that the sign of this third derivative is negative exactly when the inequality

$$\frac{k^{(1)}(x) + w_{\text{low}}}{\sqrt{\text{Var}_1(n_1)}} > \frac{1}{\sqrt{1+\beta}} - \frac{\sqrt{\text{Var}_1(n_2)}}{\mathbb{E}_1(n_2)} \tag{29}$$

holds.

On one hand, $k^{(1)}(x)$ increases without bound as resources x increase, so that for wealthy households curvature is decreasing and the third derivative is negative. This once again reflects the intuition that wealthy households behave like permanent income agents, so that the curvature in their optimal savings vanishes. On the other hand, for less wealthy households, the left hand side becomes arbitrarily small and positive as we take resources to zero while holding the properties of the risky income asset fixed, so that in the limit we obtain increasing curvature for the poorest households under condition (28).[6]

More generally, the above inequality can be interpreted in terms of a household's initial choice of asset portfolio. In particular, the right hand side depends only on the discount factor β and the risky income asset, increasing as households become less patient or the quality of the risky income asset, as measured by its Sharpe ratio, improves. Meanwhile, the left hand side is the risk free portion of this portfolio, $k^{(1)}(x) + w_{\text{low}}$, consisting of saved capital and guaranteed labor income, scaled by the standard deviation of the risky portion, and must be nonnegative.

It follows that only in cases where households are sufficiently impatient or second period income has a high risk-weighted return can the curvature in savings be increasing, and then only when a household's insurance against labor income risk is limited. These cases are those in which the household has a strong motive to behave as a hand-to-mouth consumer. However, as a consequence of the incomplete markets and the asymptote in the period utility function, exact hand-to-mouth behavior is not optimal for any household with positive wealth; such behavior opens up the possibility of encountering unemployment and having nothing to consume in the second period. Rather, when the hand-to-mouth motive is strong, the very poor households will approximate such behavior.

[6] In this simple model, the left side of this condition simplifies to $p/(1-p)$.

Indeed, mathematically the savings function (21) is a piece of a hyperbola in the (x, k) plane which has a horizontal asymptote in the limit $x \to -\infty$. As we increase the hand to mouth motive, the vertex of the hyperbola, where the curvature is the greatest, can enter the set of feasible initial resource allocations. As this vertex is further pushed to the right as, for example, the quality of second period income improves, the horizontal left hand asymptote becomes more prevalent in the set of feasible household choices, and the accuracy of the hand-to-mouth approximation increases.

Summarizing, households in the economy essentially fall into two representative groups: a hand-to-mouth representative with savings propensity near zero, and a permanent income representative with a positive savings propensity, with a region of rapid change in propensities throughout the middle class. The former class of agents only appears when the hand-to-mouth motive outweighs the certain component of future income, as will happen with impatient households or likely high second period income.

For longer horizons, a household's future decisions enter into the optimality condition for its savings function at any age $t < T - 1$. Unlike the case of rich households, passing to a limit at the borrowing constraint is not particularly helpful for controlling these additional components, so that the manipulations used to prove the above result do not generalize in a transparent way. We therefore examine the curvature of the savings functions numerically for the finite life cycle Aiyagari model described in Section 3. The household's problem was solved at the equilibrium values of the aggregate state variables using Euler equation iteration to within a convergence tolerance of 10^{-8} on a fine grid containing 10,000 grid points distributed linearly on the interval $[0, 60]$. Since we are interested in the higher order shape of the savings function, we employ a shape-preserving Schumaker spline at the interpolation stage of the iteration. This method of interpolation produces a continuously differentiable approximation which preserves local convexity and concavity in the data (see, for example, Judd (1998)).

The household savings functions obtained in the above procedure were then differentiated twice using the Julia package ForwardDiff (see Revels et al. (2016)), which uses the method of forward mode automatic differentiation. A plot of the result for the baseline model is shown in Figure 2 for households of ages $t = 1, 56, 59$. A similar plot for the the model with a high quality risky income asset ($p = 0.999$) is shown in Figure 3.

For the baseline model, the non-monotonic second derivative of savings is evident, increasing for those households with low resource holdings and decreasing for high holdings, with a maximum near the expected excess wage for an employed individual, $pw_{\text{high}} \approx 0.98$. In the case of the eldest households, $t = 59$, this observation is in line with the theory given above. Indeed, with the given parameterization we have

$$\begin{aligned}\frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)^2} &= \frac{p\ell_{\text{high}}^2 - (p\ell_{\text{high}})^2}{p^2\ell_{\text{high}}^2} \\ &= \frac{1-p}{p} \\ &\approx 0.042\end{aligned}$$

and

$$\frac{1}{1+\beta} = 0.510$$

so that the condition of Theorem 2 is easily satisfied. Moreover, the non-monotonicity is magnified for younger households relative to older households. While the region of maximum curvature shifts to the left slightly as t decreases, it remains well within the feasible set of resources. This feature is in fact apparent even in an infinite horizon analogue of the model.

Turning to the model with a better probability of high labor outcomes, we see that making this change drastically increases, but also tightens, the second derivatives of the optimal savings rules. The result is that the savings rules become essentially flat for low wealth individuals as well as high wealth individuals, so that the hand-to-mouth behavior becomes readily apparent. The curvature is much more localized but remains in essentially the same location as for the previous case, indicating a much more rapid transition from hand to mouth behavior to permanent income behavior as a household moves across the distribution of wealth from being poor relative to expected income to being relatively wealth.

Once again, it worth considering whether the results regarding savings curvature extend to environments with more general stochastic processes, utility functions, or borrowing constraints. Of these, the constraint is the easiest to consider: once again, the household Euler equation will continue to hold away from the constraint, so the result will carry over provided the non binding region overlaps the region of increasing curvature. Relaxing the IID assumption on the stochastic process for income to allow for, as an example, persistence in income states is not difficult, however additional outcomes for income introduce significant complications. Likewise, moving away from the logarithmic period utility function also complicates the derivations, although

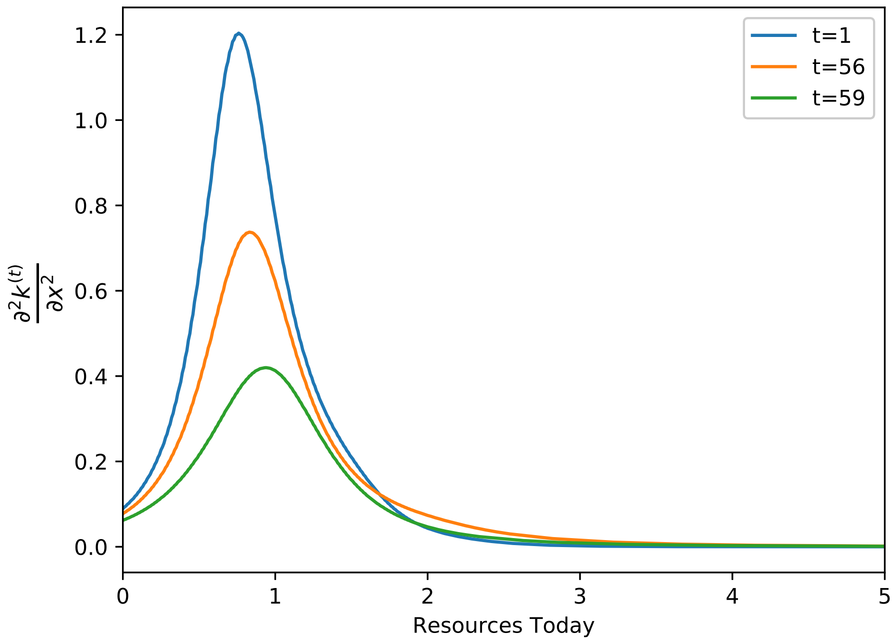

Figure 2: Curvature of the household savings function.

numerical computations exhibit some robustness to this change in specification. This being the case, we do not pursue further generality in our result for savings function curvature, having established that monotonically decreasing curvature is not observed even in the simplest of examples. Rather, we now turn to its implications for the construction of state space grids.

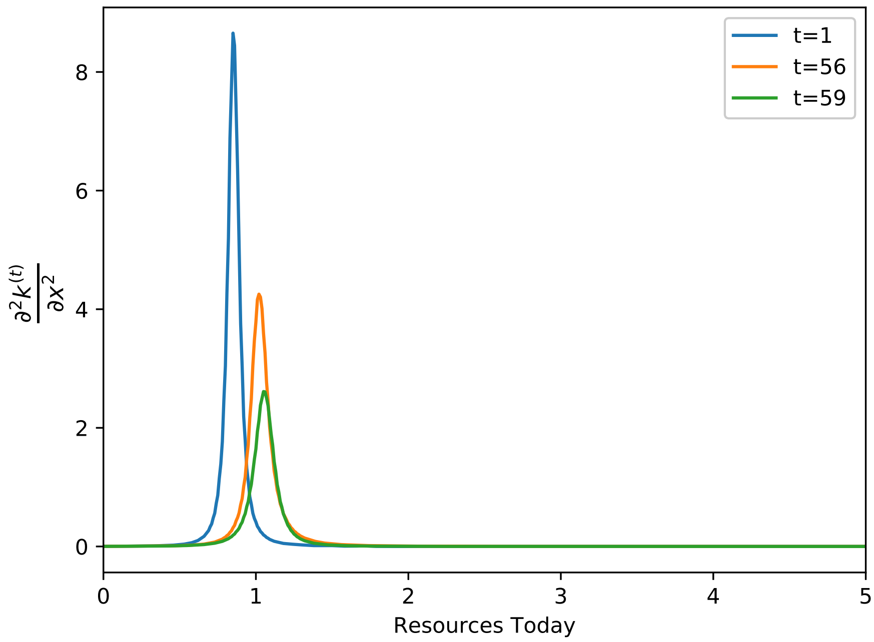

Figure 3: Curvature of the household savings function with a high quality income asset.

5 Numerical Grid Construction for Household Savings

In the previous section, it was noted that an optimal linear approximation of the savings rule used by sufficiently wealthy households will occur on grid points which become increasingly sparse as this wealth increases. This is unsurprising, as intuition and theory both indicate that such households behave increasingly like a permanent income representative with a constant propensity to save as individual wealth grows, so that the curvature in their behavior monotonically decreases. Moreover, it was noted in the previous section that this monotonicity fails for a typical calibration of a simple income fluctuation problem, in the region where household wealth is comparable to discounted expected risky income.

5.1 Variants of Algorithm 1 for Savings Functions

In this section we consider adaptations of Algorithm 1 which can be used to approximate the optimal grid for household savings, with our intent being to examine the consequences of non monotonic curvature for the shape of the optimal linear approximation across resource allocations including the poorest households. In the case of a short horizon model, with $T = 2$ and with two possible efficiency outcomes, the application is direct: the savings function is given by the closed form expression (21) which takes the place of $f(x)$ in the algorithm. We can likewise use this closed form to compute the optimal grid for approximating the savings of the age $t = 59$ households in the calibrated life cycle model, computing 30 optimal grid points in under a second on a laptop.

For $T > 2$ and $t < 59$, however, closed form expressions for the savings function are inaccessible, and a direct application of Algorithm 1 is not possible. As an alternative, we perform the following procedure to approximate the optimal grid for younger households by combining this algorithm with iterating on the Euler equation.

Algorithm 2: Euler Equation Iteration with Gavrilović grids.

1) Calculate the optimal grid $(\boldsymbol{x}^{(59)}, \boldsymbol{k}^{(59)})$ for $t = 59$ using Algorithm 1 directly.

2) Interpolate[7] the data $(\boldsymbol{x}^{(59)}, \boldsymbol{k}^{(59)})$ as $\tilde{k}^{(59)}$, and set $k^{(58)}(x)$ to be the solution for k in the approximate Euler equation

$$\frac{1}{x-k} = \left(\frac{\beta(1-\delta+R)p}{(1-\delta+R)k + We_{\text{high}} - \tilde{k}^{(59)}\left[(1-\delta+R)k + We_{\text{high}}\right]} \right.$$
$$\left. + \frac{\beta(1-\delta+R)(1-p)}{(1-\delta+R)k + We_{\text{low}} - \tilde{k}^{(59)}\left[(1-\delta+R)k + We_{\text{low}}\right]} \right)$$

given, for example, by some nonlinear solution algorithm.[8]

[7] In principle the interpolation method may be chosen arbitrarily, although the method is meant to approximate the optimal grid for a linear approximation. Applying the procedure while interpolating with a shape preserving spline gives similar accuracy gains as compared to applying it in with a linear approximation.

[8] Specifically, we can define a Julia function which takes the cash-on-hand x as an input and then performs a root finding procedure to determine k. This function may then be differentiated numerically in the optimality conditions for the grid selection. To perform the root finding procedure, we once again use binary search on the natural bounding interval $(0, x)$ for k.

3) Apply Algorithm 1 to calculate the Gavrilović grid $(\boldsymbol{x}^{(58)}, \boldsymbol{k}^{(58)})$ approximating $k^{(58)}(x)$.

4) Iterate on steps 2 and 3 to obtain $(\boldsymbol{x}^{(t)}, \boldsymbol{k}^{(t)})$ for $t = 57, ..., 1$.

While this method allows us to deduce some properties about optimal grids, such as the potential precision gains and shape, it comes with a number of limitations.

First, it is important to note that, due to the multiple stages of approximation in this procedure, the computed grids may not exactly coincide with the theoretical optimal grids for linearly approximating household savings, in the sense of Gavrilović. This motivates the adjustment of our terminology, referring to the result as a Gavrilović grid as opposed to an optimal grid. As will be seen in the discussion of accuracy below, the resulting approximations do provide improvements along a variety of metrics as compared to standard grid selections, suggesting that they are reasonable proxies to the theoretical optimum.

Second, Algorithm 1 relies on there being curvature in the function being approximated, whereas the curvature of the savings function diminishes for wealthy households. Combined with finite numerical precision and the various layers of approximation, this results in the possibility that continuity of the final grid point location x_{N+1} as a function of the worst approximation error V_0 may break down and the method will fail to stop. To ensure convergence is reached when applying Algorithm 1 in Step 3, we introduce the additional stopping criterion that the final gridpoint should be fixed as $x_{N+1} = b$ when the upper and lower bounds for V_0 in the binary search are sufficiently close together. This stipulation essentially approximates the wealthy savings function as being a single line, and has the potential to move us farther from the theoretical optimal grids which are being approximated, in particular if and when this final linear component is used to extrapolate beyond the upper bound for the discretized state space.

Last, and most severe, is that while Algorithm 1 is able to compute optimal grids very quickly, Algorithm 2 is subject to significant slowdown due to the additional root finding procedure in Step 2. This search needs to be performed several times when locating each grid point for a given choice of approximation error in Algorithm 1. As a result the gains in accuracy, while notable, are overwhelmed by the increase in computation time for general use. While we report these gains below, it is not proposed that this procedure necessarily be applied in favor of selecting a large number of grid points in cases where the latter is feasible.

A significantly less expensive approach is to forgo the specification of the number of grid points to be selected, and instead specify a desired tolerance for the accuracy of the approximation, as in the following variant of Algorithm 1.

Algorithm 3: Grids Constructed with a Predetermined Accuracy.

1) Fix a value for the desired approximation error $V_0 = V_0(x_n, x_{n+1})$ and the interval $[a, b]$. Let $x_0 = a$ and $y_0 = f(a) - V_0$.
2) Taking V_0 as given, solve (2)-(4) for x_{n+1} and y_{n+1} given x_n. More precisely, one can solve the reduced conditions (7) and (6) for x_{n+1} given x_n, and y_{n+1} can be determined from (4).
3) If $x_{n+1} > b$, stop. Otherwise, increment n and return to 2.

The primary advantages of this procedure are that the specification of a desired accuracy may be more in line with what a modeler is interested in, and that it removes a layer of convergence, namely that of the error V_0 to the optimal value. Removing the need to iterate on the error reduces time to compute the approximate grid for a given value of t to a few seconds, down from times on the order of minutes, depending on the various parameters around the grid selection as well as the hardware being used. Consequently, applying Algorithm 2 with Algorithm 3 replacing Algorithm 1 in Step 3 greatly mitigates the computational cost, while concerns about accuracy and robustness remain. In particular is not guaranteed that the approximation error to the theoretical function $k^{(t)}$ will in fact be within V_0, and in practice some accuracy is indeed lost.

Another option for increasing efficiency for constructing a grid is to solve the model using a standard grid and a shape preserving interpolation scheme, and then apply Algorithm 1 or Algorithm 3 to the resulting approximation. This procedure removes the bottleneck of needing to numerically solve the Euler equation at arbitrary points and produces grids with the same qualitative properties outlined below, but evidently has the disadvantage that one must solve the model prior to determining the grid. It may, however, be useful in cases where the grid is being computed for use in a subsequent calculation, for example in a perturbation method as in Reiter (2009) after having computed the model's steady state. Such applications are beyond the scope of this article, however, and we do not pursue this method further in this article.

5.2 Numerical Grid Properties and Accuracy

We now describe the qualitative and quantitative results from applying Algorithm 2 to the problem of household savings in the simple model of Section 3.

Figure 4 shows the resulting equilibrium savings functions for households of age t equal 1, 56, and 59, with markers placed at 30 Gavrilović grid points $(x_i^{(t)}, k_i^{(t)})$, $i = 0, ..., 29$. In agreement with conventional wisdom as well as the theory in the previous section, visually the grid point density decreases for wealthy households, with the majority of the grid points located relatively near the borrowing constraint. As age decreases and the future horizon increases, this feature is exaggerated, and relatively more points are placed near the constraint and fewer away from it, as savings propensity increases in the wealthy limit and so does curvature near the hand-to-mouth/permanent income transition region.

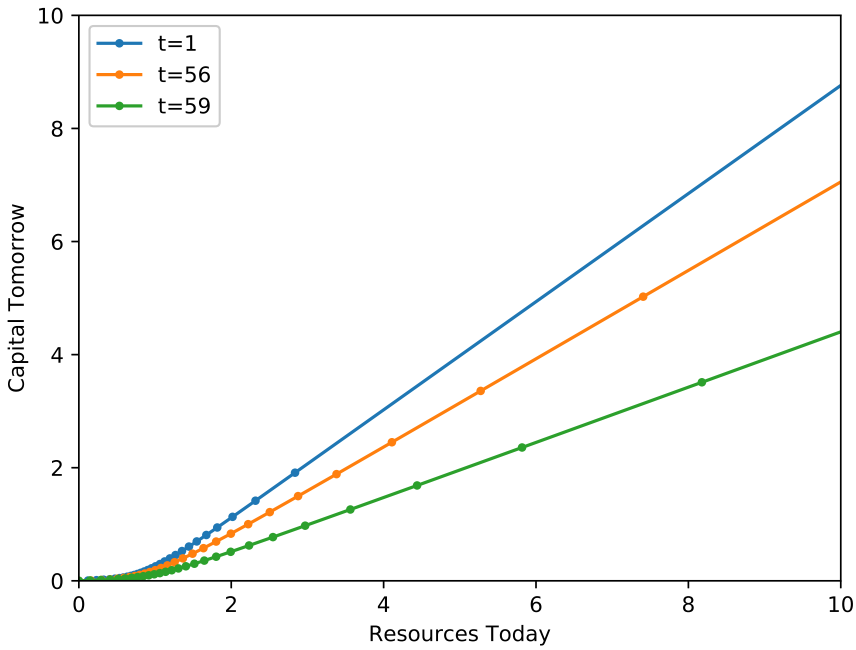

Figure 4: Equilibrium household savings functions with markers placed at Gavrilović grid points $(x_i^{(t)}, k_i^{(t)})$, for $i = 0, ..., 29$ and $t = 1, 56, 59$. The resource space has been truncated to highlight the curvature near the borrowing constraint.

On the other hand, it is not the case that grid point density is decreasing across the entire set of possible resource allocations, a feature which is not clear from a visual inspection of Figure 4. Instead, to make the point Figure 5 plots the location of the jth optimal grid point, $x_j^{(t)}$, as a function of the index j. Were it the case that grid point density decreased across the entire state space, say if $x_{j+1}^{(t)} - x_j^{(t)}$ were an increasing function of j, this curve would be convex at every index. However this is clearly not the case at indexes below $j = 10$. In addition, for younger households, the Gavrilović grid places relatively more grid points near the borrowing constraint as compared to that for older households,as the curvature of the savings function shifts to lower resource states. As a result, the non-convexity of the location index curve becomes slightly subdued, but nonetheless is still present.

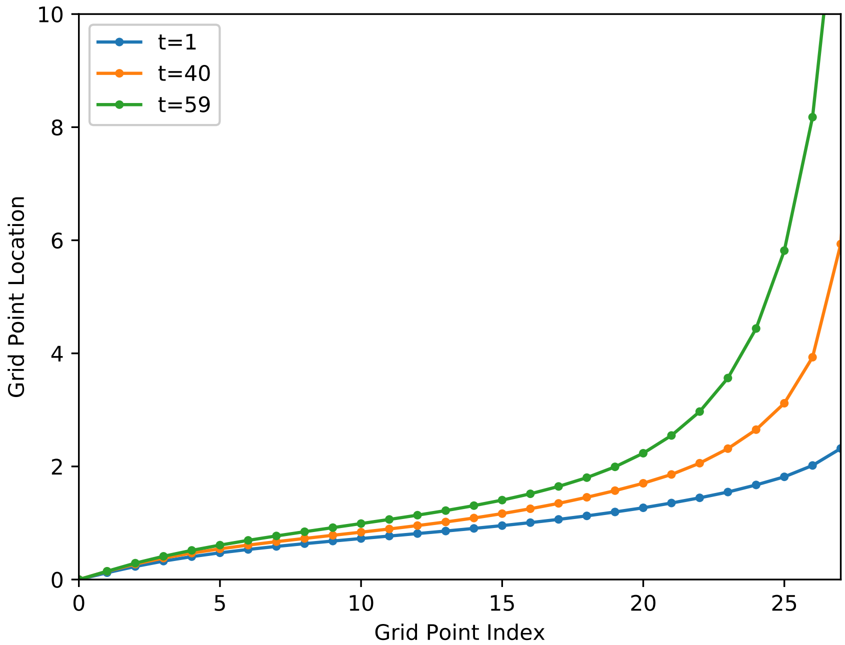

Figure 5: Location of Gavrilović grid points for the linear approximation of the equilibrium household savings functions, with the index space truncated to highlight the changing curvature.

Following the theory for how the quality of risky income impacts the hand-to-mouth motive of the households in the model from the previous section, we next consider the optimal grids for the savings function in the case in which the likelihood of high second period income is increased to $p = 0.999$.

A comparison of the resulting savings decision for a young ($t = 1$) household in this case and the baseline model is given in Figure 6. In line with the much more localized curvature in Figure 3 relative to Figure 2, we see that with a high quality income asset the savings decision is essentially linear outside of a very localized region away from the borrowing constraint, as both the hand-to-mouth and permanent income representative approximations become evident.

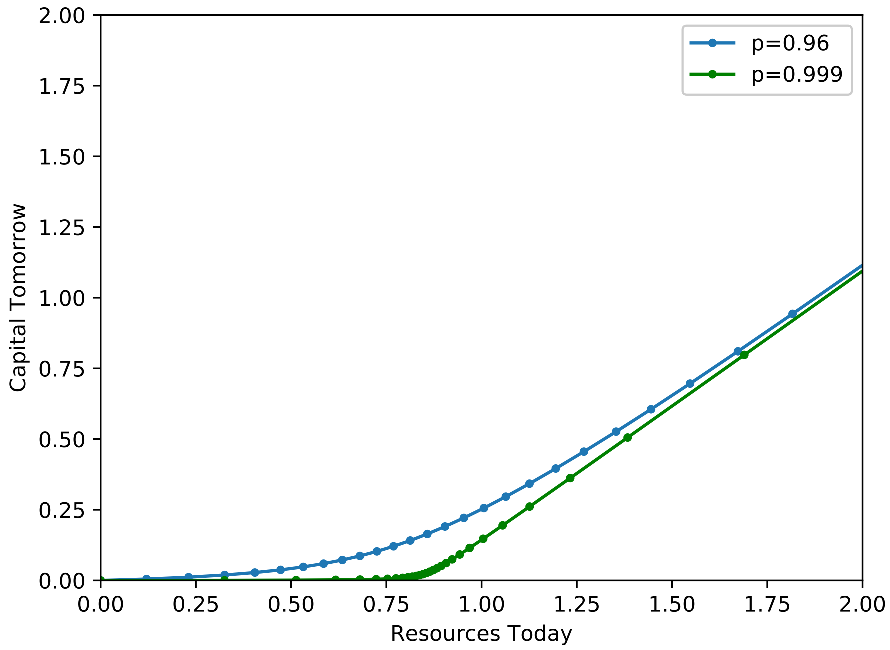

Figure 6: Comparison of the initial ($t = 1$) equilibrium household savings function with the baseline income asset ($p = 0.96$) and the high quality asset ($p = 0.999$). Markers are placed at optimal grid points $(x_i^{(t)}, k_i^{(t)})$, $i = 0, ..., 29$. The resource space has been truncated to highlight the difference in curvature between the two functions.

Consequently, the increasing grid point density in the hand-to-mouth region is transparently visible in the savings function with the high quality income asset in Figure 6. Once again plotting optimal grid point location as a function of its index in Figure 7 further emphasizes the point, demonstrating a clear concave region at low indexes followed by a nearly flat curve for indexes at which the grid points are closely clustered, as happens at the transition between the representative approximations. Once again we note that these features are evident for agents of all ages.

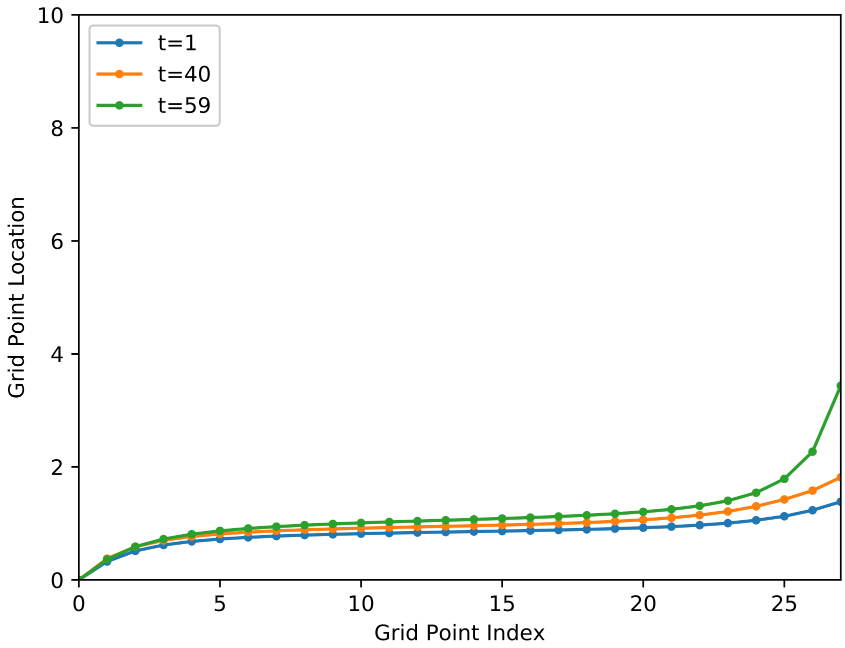

Figure 7: Location of Gavrilović grid points for the linear approximation of the equilibrium household savings functions with the high quality income asset, with the index space truncated to highlight the changing curvature..

Finally, we turn to quantifying the accuracy of a linear approximation of the savings decisions of households in the equilibrium economy using the Gavrilović grids computed as above as compared to more conventional approaches. There are a number of measures by which one might evaluate this accuracy. First, we might ask whether the approximation is in fact close to the true mathematical savings function. The latter being unknown to us, hence the need for numerical methods in the first place, we instead evaluate the accuracy of approximations on sparse grids (of size 30 and 60 points) relative to the linear approximation obtained on a very fine grid (of size 10,000 points).

While the Gavrilović grid is constructed with the objective of minimizing maximum absolute error, for practical purposes we may be more interested in some alternative measure of accuracy. For this reason, we consider the effectiveness of the grid along a variety of dimensions, including the maximum absolute error (Max), the mean absolute error (Mean), the maximum error relative to the fine grid approximation value (Max Rel.), the mean relative error (Mean Rel.), and finally the mean squared error (Mean Sq.). In addition, recalling that in general equilibrium accuracy is most crucial in those parts of the state space which contain a large fraction of the households at each age, we further construct density weighted versions of the mean statistics, respectively labeled as DWM, DWM Rel., and DWM Sq.

The results of the above accuracy tests are given in Table 1 for the baseline parameters as well as for the model with high likelihood of high labor income. In addition to the Gavrilović grids, accuracy was evaluated for three standard grid types: Linear, Logarithmic (Base 10), and Polynomial (Degree 5).[9] In particular, the jth grid points of these standard types are given, respectively, by

$$x_j = a + \frac{j}{N-1}(b-a)$$
$$x_j = 10^{\log_{10}(a)+\frac{j}{N-1}(\log_{10}(b)-\log_{10}(a))}$$
$$x_j = a + \left(\frac{j-1}{N-1}\right)^5 (b-a)$$

for $j = 1, ..., N-1$ where N is the number of grid points and a and b are the left and right hand endpoints.

Initially we observe that the Gavrilović grids perform noticeably better than the alternative schemes for the dimension along which they are chosen, namely targeting the maximum absolute error. Specifically, for the baseline model the procedure implementing the optimization algorithm is more accurate by a factor of about 10 for the smaller grid size and 6 for the larger grid size than the next best, which is the Polynomial grid. With a strong likelihood of high income, the result is sharpened to factors of about 66 and 110 respectively.

Looking further into the table, we see that for all measures apart from the mean absolute difference the Gavrilović grid provides the minimal error, where in the case of the mean difference the Polynomial grid does slightly better. Apart from this detail, there is a clear ordering of the various schemes: the Linear grid is the least accurate, followed by the Logarithmic grid, then Polynomial, then Gavrilović. Across the board accuracy increases with the number of grid points, as expected, while approximately preserving the ordering of the schemes and their relative performance.

[9] We do not provide a comparison with the accuracy obtained using Carroll's endogenous grid method. As was discussed in the introduction, this method also requires the selection of a fixed, exogenous grid, which could be any of the options considered here, or any other particular specification.

p = 0.96								
Grid Type	Linear		Logarithmic		Polynomial		Gavrilović	
n	30	60	30	60	30	60	30	60
Max	0.36	0.13	0.13	4.74(-2)	2.43(-2)	6.23(-3)	6.25(-3)	6.40(-4)
Mean	1.25(-2)	3.22(-3)	9.47(-3)	1.62(-3)	8.40(-4)	2.10(-4)	1.72(-3)	4.45(-4)
Max Rel.	14.00	6.86	1.09	0.32	0.16	3.86(-2)	0.12	2.39(-2)
Mean Rel.	0.14	5.58(-2)	1.55(-2)	4.23(-3)	2.22(-3)	5.45(-4)	4.03(-4)	6.42(-5)
Mean Sq.	2.69(-3)	2.27(-4)	4.29(-4)	3.60(-5)	9.05(-6)	5.69(-7)	5.11(-6)	5.36(-8)
DWM	1.86(-5)	5.00(-6)	8.42(-6)	2.37(-6)	1.14(-6)	2.87(-7)	1.68(-7)	1.35(-8)
DWM Rel.	1.86(-4)	7.84(-5)	2.34(-5)	6.56(-6)	3.44(-6)	8.28(-7)	3.46(-7)	5.87(-8)
DWM Sq.	4.34(-6)	3.86(-7)	6.52(-7)	6.09(-8)	1.55(-8)	9.44(-10)	3.95(-10)	1.91(-12)
p = 0.999								
Grid Type	Linear		Logarithmic		Polynomial		Gavrilović	
n	30	60	30	60	30	60	30	60
Max	0.46	0.12	0.17	8.45(-2)	6.26(-2)	2.26(-2)	9.15(-4)	5.98(-4)
Mean	1.07(-2)	1.35(-3)	4.36(-3)	6.40(-4)	3.23(-4)	7.21(-5)	2.86(-4)	1.91(-4)
Max Rel.	550.84	162.26	13.35	5.95	5.71	1.36	0.46	0.20
Mean Rel.	4.68	0.99	7.08(-2)	2.18(-2)	1.80(-2)	3.30(-3)	1.60(-3)	4.47(-4)
Mean Sq.	3.08(-3)	7.40(-5)	2.37(-4)	2.40(-5)	1.09(-5)	7.43(-7)	1.26(-7)	5.75(-8)
DWM	1.50(-5)	2.20(-6)	3.80(-6)	7.18(-7)	4.51(-7)	8.95(-8)	6.39(-8)	3.66(-8)
DWM Rel.	6.64(-3)	1.83(-3)	9.34(-5)	2.74(-5)	2.12(-5)	3.89(-6)	1.94(-6)	5.20(-7)
DWM Sq.	4.50(-6)	1.51(-7)	2.44(-7)	2.20(-8)	1.03(-9)	9.44(-10)	3.78(-11)	1.44(-11)

Table 1: Errors in linear approximations of the savings functions. Orders of magnitude are given in parentheses.

The relative ordering of the various grid types is in broad agreement with the ad-hoc experiment of Maliar et al. (2010), in which the authors examined the numerical accuracy of the solution of the Real Business Cycle model with aggregate and idiosyncratic uncertainty of Krusell & Smith (1998) for various standard grids, settling on a Polynomial grid. Comparing our standard grid schemes with the Gavrilović optimum sheds some light on why a similar observation holds in our environment.

Figure 8 reproduces Figure 5 showing the locations of the grid points in relation to their indexes for the Gavrilović grids, while also including the three standard specifications. We see from this that the Logarithmic and Linear grids are essentially two extremes, with the former placing virtually all of its grid points earlier than the Gavrilović grids and the latter placing them all later. The Polynomial grid, meanwhile, makes the former error near the borrowing constraint and the latter away from it. Mixing the two extremes allows it to remain close to the optimal layout, relative to these extremes.

One downside to the above comparison is its reliance on having already calculated a proxy for the true savings function, calling into question both the accuracy of the proxy as well as the applicability of such an assessment

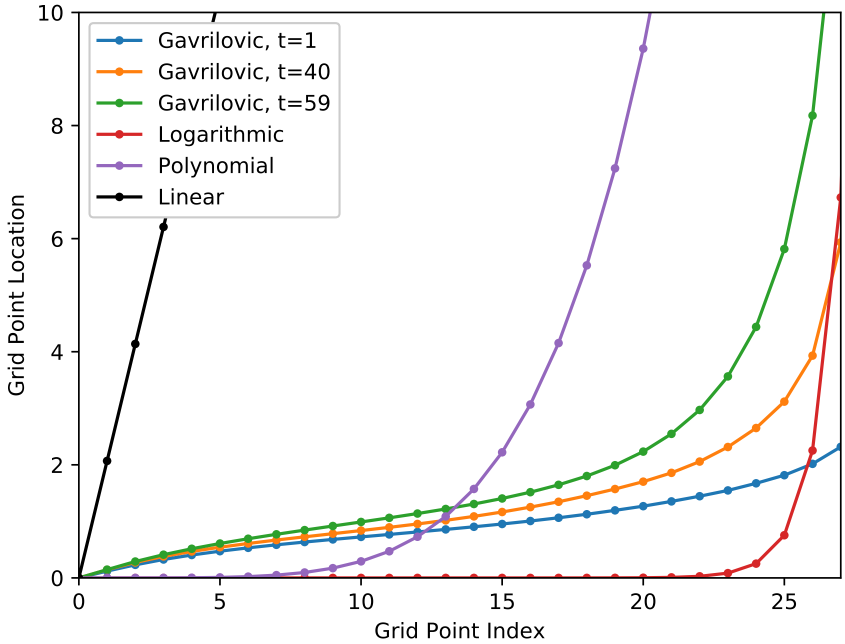

Figure 8: Location of Gavrilović grid points for the linear approximation of the equilibrium household savings functions.

mechanism for a model in which such a proxy is not easily obtained. An alternative scheme for evaluating accuracy, requiring no such proxy, is to instead determine how well the approximation satisfies the households' conditions of optimality, in this case measured by the Euler equation residuals. Table 2 reports these residuals as statistics analogous to those in Table 1, with the relative values now being given by the residual as a fraction of current consumption.

For the baseline model, the results are essentially identical to our initial assessment of the relative accuracies, with the Gavrilović grid outperforming the Polynomial grid in all but the maximum relative error, and with the Polynomial grid outperforming the Logarithmic grid which in turns outperforms the Linear one. For the high quality income asset, the results are modified slightly, with the Polynomial grid slightly outdoing the Gavrilović grid in terms of the density weighted mean relative residuals as well. However, the differences for these two statistics are fairly small, and the case for the Gavrilović grid remains quite strong.

p = 0.96								
Grid Type	Linear		Logarithmic		Polynomial		Gavrilović	
n	30	60	30	60	30	60	30	60
Max	0.61	0.32	0.21	6.42(-2)	3.24(-2)	9.10(-3)	8.59(-3)	1.26(-3)
Mean	3.33(-2)	1.15(-2)	2.38(-2)	7.02(-3)	2.79(-3)	7.13(-4)	2.24(-3)	8.81(-5)
Max Rel.	8.72	5.02	0.38	9.73(-2)	6.10(-2)	1.44(-2)	0.11	2.19(-2)
Mean Rel.	6.14(-2)	2.64(-2)	1.38(-2)	3.97(-3)	1.75(-3)	4.42(-4)	1.13(-3)	6.62(-5)
Mean Sq.	3.40(-3)	1.10(-3)	8.10(-4)	7.60(-5)	1.50(-5)	9.60(-7)	6.17(-6)	1.41(-8)
DWM	1.62(-5)	7.18(-6)	6.50(-6)	2.05(-6)	9.80(-7)	2.46(-7)	3.31(-7)	1.53(-8)
DWM Rel.	7.08(-5)	3.16(-5)	9.15(-6)	2.65(-6)	1.36(-6)	3.37(-7)	4.17(-7)	3.36(-8)
DWM Sq.	3.45(-6)	1.39(-6)	3.88(-7)	4.59(-8)	1.26(-8)	8.04(-10)	7.89(-10)	2.54(-12)
p = 0.999								
Grid Type	Linear		Logarithmic		Polynomial		Gavrilović	
n	30	60	30	60	30	60	30	60
Max	1.03	0.74	0.30	0.19	0.17	5.68(-2)	3.77(-2)	1.17(-2)
Mean	3.07(-2)	9.57(-3)	1.21(-2)	3.58(-3)	1.41(-3)	3.04(-4)	6.42(-4)	3.54(-4)
Max Rel.	26.58	17.12	0.89	0.32	0.22	6.78(-2)	0.43	0.18
Mean Rel.	8.52(-2)	4.81(-2)	9.29(-3)	2.80(-3)	1.48(-3)	3.53(-4)	1.41(-3)	4.69(-4)
Mean Sq.	5.39(-3)	4.40(-3)	5.41(-4)	1.05(-4)	7.01(-5)	6.29(-6)	3.97(-6)	3.48(-7)
DWM	1.89(-5)	1.13(-5)	5.05(-6)	1.87(-6)	1.14(-6)	3.05(-7)	2.62(-7)	8.87(-8)
DWM Rel.	1.09(-4)	6.83(-5)	8.71(-6)	2.79(-6)	1.70(-6)	4.46(-7)	1.50(-6)	4.11(-7)
DWM Sq.	5.92(-6)	5.79(-6)	6.42(-7)	1.45(-7)	7.12(-8)	7.30(-9)	3.59(-9)	2.15(-10)

Table 2: Euler equation errors in linear approximations of the savings functions. Orders of magnitude are given in parentheses.

Finally, Table 3 displays the error metrics obtained from repeating the accuracy exercise using Algorithm 3 instead of Algorithm 1 in Algorithm 2, with two different specified tolerances. We first observe that the maximum absolute error is approximately an order of magnitude worse than the tolerance specified, owing to our explanation in Section 5.1 that We also observe that reducing the tolerance by an order of magnitude increases accuracy by a similar factor in most cases, while only increasing the grid size by a factor of about 3, averaged across the life cycle in the model. Lastly, increasing the income asset quality roughly halves the number of grid points needed, likely due to the region of curvature being narrower in this instance.

	p=0.96		p=0.999	
Error Tolerance	10^{-4}	10^{-5}	10^{-4}	10^{-5}
Ave. N	40	123	21	63
Max	1.31(-3)	1.13(-4)	1.54(-3)	1.51(-4)
Mean	2.60(-4)	1.17(-5)	6.05(-4)	4.75(-5)
Max Rel.	5.31(-2)	5.28(-3)	0.72	0.16
Mean Rel.	1.47(-4)	1.60(-5)	3.17(-3)	3.99(-4)
Mean Sq.	1.53(-7)	3.75(-10)	5.04(-7)	3.81(-9)
DWM	2.85(-8)	3.34(-9)	2.90(-8)	2.17(-9)
DWM Rel.	1.79(-7)	1.96(-8)	4.27(-6)	5.40(-7)
DWM Sq.	8.11(-12)	1.16(-13)	9.54(-12)	6.53(-14)

Table 3: Errors in linear approximations of the savings functions constructed in each Euler equation iteration to a desired approximation error. Orders of magnitude are given in parentheses.

6 Conclusion

Interpolation of discrete approximations to convex functions has long been a feature of numerical methods in economics. The theory and numerical procedure of Gavrilović (1975) provides a method by which we can optimize this procedure. Applied to the context of savings decisions by a household in a simple life cycle economy, we obtain that the optimal approximation becomes sparse as wealth increases, in accordance with the conventional understanding that household behavior is approximately that of a representative, linear agent in this limit. Less expectedly, the trend of increasing sparseness reverses in the low wealth region for a typical selection of model parameters, as the region of maximal curvature in the savings decision lies interior to the resource state space. The accuracy gains from using the Gavrilović approximation are substantial as compared to those obtained using various more standard methods.

There remains a number of worthwhile questions to be addressed in future literature. First, a substantial contribution would be to construct an efficient method for determining an optimal grid for a function given implicitly by a set of conditions such as (20), without succumbing to the curse of dimensionality or relying on prior approximations as we did in Algorithm 2. As well, the results provided by Gavrilović (1975) rely closely on the assumption that the function being approximated is univariate. In particular, Algorithm 1 requires that the grid points be selected in increasing order. However, many economic problems are inherently higher dimensional, which raises the issue of how best to adapt these ideas to address such problems. Since there is no clear ordering of grid points in a multidimensional state space, this appears to be a challenging issue to resolve. One approach which may be tractable in simple cases[10] is to simply divide the problem into multiple univariate optimizations. A somewhat related question is how one might optimize the grid selection with alternative interpolation schemes in mind, such as the Schumaker splines used here or cubic splines.

There also remain outstanding theoretical questions regarding the curvature of the savings function and its relationship to the grid selection. Theorem 2 establishes a clean result for the non monotonicity of household savings in a simple, static environment, with vivid intuition in terms of a hand-to-mouth representative agent for the resource poor in the economy alongside the long established permanent income representative for the resource wealthy. Figures 2 and 3 show that this feature persists in dynamic models, raising the likely prospect that it appears in some form in more complicated frameworks as well. Further results in this direction would be welcome. In particular, placing the connection between curvature and grid density on a solid theoretical foundation might allow for a more efficient determination of optimal grids, insofar as they could be determined simply by numerical differentiation rather than a more costly optimization.

[10] For example, in the case of correlation between the shocks in our life cycle model.

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A Proofs

Here we provide the proofs of our main results. We will begin by giving a short outline of the derivation of (23) in the setting of this paper, with log utility and IID income with two outsomes. The full argument is given for general CRRA utility and stochastic income processes in Chipeniuk et al. (2016).

Proof of (23)**, Outline, Special Case:** The essential elements of the proof are contained in the case $t = 2$, where the necessary optimality condition (20) can be written as

$$\frac{1}{x_1 - k^{(1)}(x_1)} = \beta \left(\frac{p}{k^{(1)}(x_1) + w_{\text{high}}} + \frac{1-p}{k^{(1)}(x_1) + w_{\text{low}}} \right) \tag{30}$$

where the expected value has been expanded out using our specification of the income process. Some algebra rewrites this as

$$\begin{aligned}
\beta(x_1 - k^{(1)}(x_1)) &= \frac{k^{(1)}(x_1)^2 + (w_{\text{high}} + w_{\text{low}})k^{(1)}(x_1) + w_{\text{high}}w_{\text{low}}}{k^{(1)}(x_1) + pw_{\text{low}} + (1-p)w_{\text{high}}} \\
&= k^{(1)}(x_1) + \frac{(pw_{\text{high}} + (1-p)w_{\text{low}})k^{(1)}(x_1) + w_{\text{high}}w_{\text{low}}}{k^{(1)}(x_1) + pw_{\text{low}} + (1-p)w_{\text{high}}} \\
&= k^{(1)}(x_1) + (pw_{\text{high}} + (1-p)w_{\text{low}}) \\
&\quad + \frac{w_{\text{high}}w_{\text{low}} - (pw_{\text{high}} + (1-p)w_{\text{low}})(pw_{\text{low}} + (1-p)w_{\text{high}})}{k^{(1)}(x_1) + pw_{\text{low}} + (1-p)w_{\text{high}}} \\
&= k^{(1)}(x_1) + (pw_{\text{high}} + (1-p)w_{\text{low}}) \\
&\quad - \frac{p(1-p)(w_{\text{high}} - w_{\text{low}})^2}{k^{(1)}(x_1) + pw_{\text{low}} + (1-p)w_{\text{high}}}
\end{aligned}$$

Rearranging, this gives

$$\begin{aligned}
k^{(1)}(x_1) &= \frac{\beta}{1+\beta}x_1 - \frac{1}{1+\beta}(pw_{\text{high}} + (1-p)w_{\text{low}}) \\
&\quad + \left(\frac{1}{1+\beta}\right) \frac{p(1-p)(w_{\text{high}} - w_{\text{low}})^2}{k^{(1)}(x_1) + pw_{\text{low}} + (1-p)w_{\text{high}}}
\end{aligned}$$

Recalling the definition of n_2 and the stochastic process for ℓ_2, we can rewrite this as

$$\begin{aligned}
k^{(1)}(x_1) &= \frac{\beta}{1+\beta}x_1 - \frac{1}{1+\beta}\mathbb{E}_1 w_2 \\
&\quad + \left(\frac{1}{1+\beta}\right) \frac{\text{Var}_1(n_2)}{k^{(1)}(x_1) + w_{\text{low}} + \frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)}}
\end{aligned} \tag{31}$$

The trailing term here is the nonlinear contribution $\epsilon^{(1)}(x_1)$, which is well-defined and well-behaved since it is known that there exists a unique, well behaved solution $k^{(1)}(x_1)$ to the household's problem (see Chipeniuk et al. (2016), Proposition 1 for details). In particular, from the properties of $k^{(1)}(x_1)$ it is seen that the nonlinear term is convex and decreases to 0 for large resource allocations.

To extend this to the dynamic setting $1 \leq t \leq T-1$, one can proceed by induction. In particular, we begin from the base case $t = T-1$, which proceeds identically to the $T = 2$ case given above. Assuming that we have proven the result for the savings function for a household of age $t+1$, we can the use the inductive hypothesis to rewrite (20) in terms of $k^{(t)}(x_t)$ and $\epsilon^{(t+1)}(x_{t+1})$. Performing a calculation similar to that given above, we arrive at the linear plus nonlinear expansion, and the properties of the nonlinear term $\epsilon^{(t)}$ follow from those of $\epsilon^{(t+1)}$, once again appealing to the inductive hypothesis. For details, see the proof of Lemma 2 below and Chipeniuk et al. (2016). □

Proof of Theorem 1, T=2: We next complete the argument which proves Theorem 1 in full for the case $T = 2$, before giving as a lemma the induction which is the primary complication in extending the method to several periods. With this lemma established, the full theorem will follow simply by repeating the two period horizon argument.

For notational simplicity we will omit time subscripts. Specifically, let $x := x_1$ denote initial resources in this setting, and let $\epsilon(x) := \epsilon^{(1)}(x)$ denote the nonlinear term in the corresponding savings function $k(x) := k^{(1)}(x)$. Recall that from (31) we have

$$\epsilon(x) = \left(\frac{1}{1+\beta}\right) \frac{\mathrm{Var}_1(n_2)}{k(x) + w_{\mathrm{low}} + \frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)}}$$

Let $D = \mathrm{Var}_1(n_2)/(1+\beta)$. Differentiating, we get

$$\begin{aligned}
\frac{\partial \epsilon}{\partial x}(x) &= \frac{-D}{\left(k(x) + w_{\mathrm{low}} + \frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^2} \frac{\partial k}{\partial x}(x) \\
&= \frac{-D}{\left(k(x) + w_{\mathrm{low}} + \frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^2} \left(A + \frac{\partial \epsilon}{\partial x}(x)\right)
\end{aligned}$$

which solves to give

$$\frac{\partial \epsilon}{\partial x}(x) = \frac{-AD}{\left(k(x) + w_{\text{low}} + \frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^2 + D}$$

Let $C = w_{\text{low}} + \frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)}$, so that we have

$$\begin{aligned}\epsilon(x) &= \frac{D}{k(x) + C} \\ \frac{\partial \epsilon}{\partial x}(x) &= \frac{-AD}{(k(x) + C)^2 + D}\end{aligned}$$

Writing

$$k(x) = Ax - B + \epsilon(x) = Ax - B + o(1) = Ax(1 + O(1/x))$$

we get

$$\begin{aligned}\epsilon(x) &= \frac{D}{Ax(1 + O(1/x))} \\ \frac{\partial \epsilon}{\partial x}(x) &= \frac{-AD}{A^2x^2\left(1 + O(1/x)\right)^2 + D} \\ &= \frac{-AD}{A^2x^2\left((1 + O(1/x))^2 + O(1/x)\right)} \\ &= \frac{-D}{Ax^2\left((1 + O(1/x))^2 + O(1/x)\right)} \\ &= \frac{-D}{Ax^2\left(1 + O(1/x)\right)}\end{aligned}$$

For initial resources x sufficiently large, the $O(1/x)$ terms in the above expressions will be smaller than 1, so that we can write[11]

$$\frac{1}{1+O(1/x)} = 1 + O(1/x) + O(1/x^2) + ... = 1 + O(1/x) \tag{32}$$

giving

$$\epsilon(x) = \frac{D}{Ax}(1 + O(1/x)) \tag{33}$$

$$\frac{\partial \epsilon}{\partial x}(x) = \frac{-D}{Ax^2}(1 + O(1/x)) \tag{34}$$

Using these we can write (25) as

$$\frac{-D}{A(x^*)^2}(1 + O(1/x^*)) = \frac{\frac{D}{Ax_{n+1}}(1 + O(1/x_{n+1})) - \frac{D}{Ax_n}(1 + O(1/x_n))}{x_{n+1} - x_n}$$

Now, since $x_n \leq x^* \leq x_{n+1}$, we have $O(1/x^*) = O(1/x_n)$ and likewise for x_{n+1}. Using this and canceling like factors, the last equation can be rewritten as

$$\frac{1}{(x^*)^2}(1 + O(1/x_n)) = \frac{\frac{1}{x_n} - \frac{1}{x_{n+1}}}{x_{n+1} - x_n}(1 + O(1/x_n))$$

For x_n sufficiently large, from a similar geometric series argument to that in the footnote above we get

$$\frac{1 + O(1/x_n)}{1 + O(1/x_n)} = 1 + O(1/x_n)$$

[11] Specifically, let $C > 0$ be a constant so that $|O(1/x)| \leq C/x$. Then for x sufficiently large we have

$$\begin{aligned} \left|\frac{1}{1+O(1/x)}\right| &\leq 1 + \frac{C}{x} + \left(\frac{C}{x}\right)^2 + ... \\ &= 1 + \frac{C}{x}\left(1 + \frac{C}{x} + \left(\frac{C}{x}\right)^2 + ...\right) \\ &= 1 + \frac{C}{x}\left(\frac{1}{1 - C/x}\right) \\ &\leq 1 + 2C/x \\ &= 1 + O(1/x) \end{aligned}$$

Then we obtain

$$\frac{1}{(x^*)^2} = \frac{\frac{1}{x_n} - \frac{1}{x_{n+1}}}{x_{n+1} - x_n}(1 + O(1/x_n))$$

so

$$x^* = \sqrt{\frac{(x_{n+1} - x_n)\,(1 + O(1/x_n))}{\frac{1}{x_n} - \frac{1}{x_{n+1}}}}$$

Applying the above calculations in (25), we obtain as an expression for $2V_0(x_n, x_{n+1})$

$$\frac{D}{Ax_n}(1 + O(1/x_n))$$
$$-\frac{\frac{D}{Ax_n} - \frac{D}{Ax_{n+1}}}{x_{n+1} - x_n}(1 + O(1/x_n))\left(\sqrt{\frac{(x_{n+1} - x_n)\,(1 + O(1/x_n))}{\frac{1}{x_n} - \frac{1}{x_{n+1}}}} - x_n\right)$$
$$-\frac{D}{A}\sqrt{\frac{\left(\frac{1}{x_n} - \frac{1}{x_{n+1}}\right)(1 + O(1/x_n))}{x_{n+1} - x_n}}(1 + O(1/x_n))$$

or

$$2V_0(x_n, x_{n+1}) = \frac{D}{A}(1 + O(1/x_n)) \times \left(\frac{1}{x_n} - \frac{\frac{1}{x_n} - \frac{1}{x_{n+1}}}{x_{n+1} - x_n}\left(\sqrt{\frac{x_{n+1} - x_n}{\frac{1}{x_n} - \frac{1}{x_{n+1}}}} - x_n\right) - \sqrt{\frac{\frac{1}{x_n} - \frac{1}{x_{n+1}}}{x_{n+1} - x_n}}\right)$$

From the calculations in Example 1, the trailing factor here simplifies to give

$$V_0(x_n, x_{n+1}) = \frac{D}{2A}(1 + O(1/x_n))\left(\frac{x_{n+1} - x_n}{x_n x_{n+1}}\right)$$

Now, applying the fact that $V_0(x_n, x_{n+1}) \equiv V_0$ for $n = 0, ..., N$, along with the fact that $O(1/x_n) = O(1/x_0)$, we can sum the above expressions over n to get

$$(N + 1)V_0 = \frac{D}{2A}(1 + O(1/x_0))\left(\frac{1}{x_0} - \frac{1}{x_{N+1}}\right)$$

Then x_1 satisfies

$$\frac{D}{2A}(1 + O(1/x_0))\left(\frac{1}{x_0} - \frac{1}{x_1}\right) = \frac{D}{2(N + 1)A}(1 + O(1/x_0))\left(\frac{1}{x_0} - \frac{1}{x_{N+1}}\right)$$

so that

$$x_1 = \frac{1 + O(1/x_0)}{\left(\frac{N}{N+1}\right)\frac{1}{x_0} + \left(\frac{1}{N+1}\right)\frac{1}{x_{N+1}}}$$

and repeating the argument from Example 1

$$x_j = \frac{1 + O(1/x_0)}{\left(\frac{N+1-j}{N+1}\right)\frac{1}{x_0} + \left(\frac{j}{N+1}\right)\frac{1}{x_{N+1}}}$$

which is the desired result for $T = 2$.

Lemma 2: Error Term Derivatives. There are constants $C^{(t)}$ such that the nonlinear terms $\epsilon^{(t)}$ in the linear-plus-error expansion of the savings functions $k^{(t)}$ which solve the household's dynamic programming problems (19) satisfy

$$\epsilon^{(t)}(x) = \frac{C^{(t)}}{x}(1 + O(1/x))$$
$$\frac{\partial \epsilon^{(t)}(x)}{\partial x} = \frac{-C^{(t)}}{x^2}(1 + O(1/x))$$

Proof. The proof is by induction, with (33) and (34) forming the base case in period $t = T - 1$. Hence we begin by assuming that the lemma has been proved for the nonlinear terms in periods $t + 1, t + 2, ..., T - 1$, $t \geq 1$.

We must begin by extracting an explicit expression for the time t error in terms of the time $t + 1$ error. Writing out the time t intertemporal Euler equation (20), we have

$$\frac{1}{x_t - k^t(x_t)} = \beta \left(\frac{p}{k^{(t)}(x_t) + w_{\text{high}} - \frac{k^{(t+1)}(x_{\text{high}})}{R_{t+1}}} + \frac{1-p}{k^{(t)}(x_t) + w_{\text{low}} - \frac{k^{(t+1)}(x_{\text{low}})}{R_{t+1}}} \right)$$

where $x_{\text{high}} = R_{t+1}k^t(x_t) + W_{t+1}\ell_{\text{high}}$ and similarly for x_{low}.

Recalling (22), we have

$$k^{(t+1)}(x_{\text{high}}) = \frac{\beta + ... + \beta^{T-t-1}}{1 + \beta + ... + \beta^{T-t-1}}(R_{t+1}k^{(t)}(x_t) + R_{t+1}w_{\text{high}})$$
$$- \frac{1}{1 + \beta + ... + \beta^{T-t-1}}\mathbb{E}_{t+1}\left(\sum_{s=t+2}^{T} \frac{W_s \ell_s}{\prod_{r=t+2}^{s} R_r}\right) + \epsilon^{(t+1)}(x_{\text{high}})$$

so that the expression $k^{(t)}(x_t) + w_{\text{high}} - \frac{k^{(t+1)}(x_{\text{high}})}{R_{t+1}}$ may be rewritten as

$$\begin{aligned} &\frac{1}{1+\beta+...+\beta^{T-t-1}}(k^{(t)}(x_t) + w_{\text{high}}) \\ &- \left(\frac{1}{R_{t+1}}\right)\frac{1}{1+\beta+...+\beta^{T-t-1}}\mathbb{E}_{t+1}\left(\sum_{s=t+2}^{T}\frac{W_s\ell_s}{\prod_{r=t+2}^{s}R_r}\right) \\ &-\frac{\epsilon^{(t+1)}(x_{\text{high}})}{R_{t+1}} \end{aligned}$$

Letting

$$\begin{aligned} A := A^{(t)} &= \frac{1}{1+\beta+...+\beta^{T-t-1}} \\ B_{\text{high}} := B^{(t)}_{\text{high}} &= A\left(w_{\text{high}} - \left(\frac{1}{R_{t+1}}\right)\mathbb{E}_{t+1}\left(\sum_{s=t+2}^{T}\frac{W_s\ell_s}{\prod_{r=t+2}^{s}R_r}\right)\right) \\ \delta_{\text{high}}(x_t) := \delta^{(t)}_{\text{high}}(x_t) &= \frac{\epsilon^{(t+1)}(x_{\text{high}})}{R_{t+1}} \\ C_{\text{high}}(x_t) := C^{(t)}_{\text{high}}(x_t) &= B_{\text{high}} - \delta_{\text{high}}(x_t) \end{aligned}$$

we therefore have

$$k^{(t)}(x_t) + w_{\text{high}} - \frac{k^{(t+1)}(x_{\text{high}})}{R_{t+1}} = Ak^{(t)}(x_t) + B_{\text{high}} - \delta_{\text{high}}(x_t)$$

and similarly

$$k^{(t)}(x_t) + w_{\text{low}} - \frac{k^{(t+1)}(x_{\text{low}})}{R_{t+1}} = Ak^{(t)}(x_t) + B_{\text{low}} - \delta_{\text{low}}(x_t)$$

Some algebra will then verify that the Euler equation can be written

$$\begin{aligned} \beta(x_t - k^t(x_t)) = Ak^{(t)}(x_t) &+ \Big[pC_{\text{high}}(x_t) + (1-p)C_{\text{low}}(x_t)\Big] \\ &- \frac{p(1-p)(C_{\text{high}}(x_t) - C_{\text{low}}(x_t))^2}{Ak^{(t)}(x_t) + pC_{\text{low}}(x_t) + (1-p)C_{\text{high}}(x_t)} \end{aligned}$$

or

$$\begin{aligned} k^{(t)}(x_t) = \frac{\beta}{\beta+A}x_t &- \left(\frac{1}{\beta+A}\right)\Big[pC_{\text{high}}(x_t) + (1-p)C_{\text{low}}(x_t)\Big] \\ &+ \left(\frac{1}{\beta+A}\right)\frac{p(1-p)(C_{\text{high}}(x_t) - C_{\text{low}}(x_t))^2}{Ak^{(t)}(x_t) + pC_{\text{low}}(x_t) + (1-p)C_{\text{high}}(x_t)} \end{aligned}$$

As in Chipeniuk et al. (2016) the nonlinear error is thus the well-defined expression

$$\left(\frac{1}{\beta + A}\right)\left(- \Big[p\delta_{\text{high}}(x_t) + (1-p)\delta_{\text{low}}(x_t)\Big] \right.$$
$$\left. + \frac{p(1-p)(C_{\text{high}}(x_t) - C_{\text{low}}(x_t))^2}{Ak^{(t)}(x_t) + pC_{\text{low}}(x_t) + (1-p)C_{\text{high}}(x_t)}\right) \tag{35}$$

To prove the lemma, we need to show that the expression in the second set of brackets satisfies the stated bounds. To begin with, from the inductive hypothesis we have

$$\delta_{\text{high}}(x_t) = \frac{C^{(t+1)}}{R_{t+1}x_h}(1 + O(1/x_h))$$

On observing that

$$\begin{aligned} x_h &= R_{t+1}k^{(t)}(x_t) + w_{\text{high}} \\ &= R_{t+1}\left(\frac{\beta + ... + \beta^{T-t}}{1 + \beta + ... + \beta^{T-t}}\right) x_t(1 + O(1/x_t)) + w_{\text{high}} \\ &= R_{t+1}\left(\frac{\beta + ... + \beta^{T-t}}{1 + \beta + ... + \beta^{T-t}}\right) x_t(1 + O(1/x_t)) \end{aligned} \tag{36}$$

where the second equality used (22) and the fact that $\epsilon^{(t)}(x_t) = o(1)$, we see that this gives

$$\delta_{\text{high}}(x_t) = \left(\frac{1 + \beta + ... + \beta^{T-t}}{\beta + ... + \beta^{T-t}}\right)\frac{C^{(t+1)}}{R_{t+1}^2 x_t}(1 + O(1/x_t))$$

after a calculation analogous to that in (32). A similar argument shows that

$$\delta_{\text{low}}(x_t) = \left(\frac{1 + \beta + ... + \beta^{T-t}}{\beta + ... + \beta^{T-t}}\right)\frac{C^{(t+1)}}{R_{t+1}^2 x_t}(1 + O(1/x_t))$$

and consequently

$$p\delta_{\text{high}} + (1-p)\delta_{\text{low}} = \left(\frac{1 + \beta + ... + \beta^{T-t}}{\beta + ... + \beta^{T-t}}\right)\frac{C^{(t+1)}}{R_{t+1}^2 x_t}(1 + O(1/x_t)) \tag{37}$$

Next we calculate

$$\begin{aligned}\frac{\partial \delta_{\text{high}}}{\partial x_t} &= \frac{1}{R_{t+1}} \frac{\partial \epsilon^{(t+1)}(x_{\text{high}})}{\partial x} \frac{\partial x_{\text{high}}}{\partial x_t} \\ &= \frac{-C^{(t+1)}}{x_{\text{high}}^2}(1 + O(1/x_{\text{high}}))R_{t+1}\frac{\partial k^{(t)}}{\partial x_t} \\ &= -\left(\frac{1+\beta+...+\beta^{T-t}}{\beta+...+\beta^{T-t}}\right)^2 \frac{C^{(t+1)}}{R_{t+1}^2 x_t^2}(1+O(1/x_t))\frac{\partial k^{(t)}}{\partial x_t}\end{aligned}$$

where the second equality uses the inductive hypothesis and the third uses (36). An identical derivation goes through for δ_{low}, so the expected value

$$p\frac{\partial \delta_{\text{high}}}{\partial x_t} + (1-p)\frac{\partial \delta_{\text{low}}}{\partial x_t}$$

can be written as

$$-\left(\frac{1+\beta+...+\beta^{T-t}}{\beta+...+\beta^{T-t}}\right)^2 \frac{C^{(t+1)}}{R_{t+1}^2 x_t^2}(1+O(1/x_t))\frac{\partial k^{(t)}}{\partial x_t} \tag{38}$$

Next, we observe that

$$\frac{p(1-p)(C_{\text{high}}(x_t) - C_{\text{low}}(x_t))^2}{Ak^{(t)}(x_t) + pC_{\text{low}}(x_t) + (1-p)C_{\text{high}}(x_t)}$$

can be expanded as

$$\begin{aligned}&\frac{p(1-p)(B_{\text{high}} - B_{\text{low}})^2}{Ak^{(t)}(x_t) + pC_{\text{low}}(x_t) + (1-p)C_{\text{high}}(x_t)} \\ &+\frac{2p(1-p)(B_{\text{high}} - B_{\text{low}})(\delta_{\text{high}}(x_t) - \delta_{\text{low}}(x_t))}{Ak^{(t)}(x_t) + pC_{\text{low}}(x_t) + (1-p)C_{\text{high}}(x_t)} \\ &+\frac{(\delta_{\text{high}}(x_t) - \delta_{\text{low}}(x_t))^2}{Ak^{(t)}(x_t) + pC_{\text{low}}(x_t) + (1-p)C_{\text{high}}(x_t)}\end{aligned}$$

It is not difficult to see, using the inductive hypothesis and arguments of the style used previously that the latter two terms are $O(1/x_t^2)$ and their derivatives are $O(1/x_t^3)$. Likewise, the first term is seen to be

$$\left(\frac{1+...+\beta^{T-t}}{\beta}\right)\frac{p(1-p)(B_{\text{high}} - B_{\text{low}})^2}{x_t}(1+O(1/x_t)) \tag{39}$$

and its derivative is equal to

$$\begin{aligned}&\frac{-p(1-p)(B_{\text{high}} - B_{\text{low}})^2}{(Ak^{(t)}(x_t) + pC_{\text{low}}(x_t) + (1-p)C_{\text{high}}(x_t))^2}\left(A\frac{\partial k^{(t)}}{\partial x_t} + p\frac{\partial \delta_{\text{low}}}{\partial x_t} + p\frac{\partial \delta_{\text{low}}}{\partial x_t}\right) \\ &= \frac{-Ap(1-p)(B_{\text{high}} - B_{\text{low}})^2}{(Ak^{(t)}(x_t) + pC_{\text{low}}(x_t) + (1-p)C_{\text{high}}(x_t))^2}(1+O(1/x_t))\frac{\partial k^{(t)}}{\partial x_t}\end{aligned}$$

This can be further simplified as

$$\frac{-p(1-p)(B_{\text{high}}-B_{\text{low}})^2}{A\left(\frac{\beta+...+\beta^{T-t}}{1+\beta+...+\beta^{T-t}}\right)^2 x_t^2}(1+O(1/x_t))\frac{\partial k^{(t)}}{\partial x_t} \tag{40}$$

Combining (37) and (39) with (35), we see that $\epsilon^{(t)}(x_t)$ is equal to $(1+O(1/x_t))$ times the expression

$$\frac{A\left(\frac{1+\beta+...+\beta^{T-t}}{\beta+...+\beta^{T-t-1}}\right)\left[-\left(\frac{1}{\beta+...+\beta^{T-t}}\right)\frac{C^{(t+1)}}{R_{t+1}^2}+\left(\frac{1}{\beta}\right)p(1-p)(B_{\text{high}}-B_{\text{low}})^2\right]}{x_t}$$

so that the first bound stated in the lemma holds with

$$C^{(t)} = A\left(\frac{1+\beta+...+\beta^{T-t}}{\beta+...+\beta^{T-t-1}}\right)\Bigg[-\left(\frac{1}{\beta+...+\beta^{T-t}}\right)\frac{C^{(t+1)}}{R_{t+1}^2} + \left(\frac{1}{\beta}\right)p(1-p)(B_{\text{high}}-B_{\text{low}})^2\Bigg]$$

Combining (38) and (40) with (35), we obtain

$$\frac{\partial\epsilon^{(t)}}{\partial x_t} = \frac{-C^{(t)}}{x_t^2}(1+O(1/x_t))\left(1+\frac{1+\beta+....+\beta^{T-t}}{\beta+....+\beta^{T-t}}\frac{\partial\epsilon^{(t)}}{x_t}\right)$$

which rearranges as

$$\begin{aligned}\frac{\partial\epsilon^{(t)}}{\partial x_t} &= \frac{-C^{(t)}}{x_t^2}(1+O(1/x_t))\frac{1}{1+\frac{C^{(t)}}{x_t^2}} \\ &= \frac{-C^{(t)}}{x_t^2}(1+O(1/x_t)) \end{aligned} \tag{41}$$

giving the second bound stated in the lemma. □

Theorem 1: Optimal Grids For Wealthy Household. The optimal grid points x_j for a linear approximation (in the sense of Gavrilović (1975)) of the savings functions $k^{(t)}$ which solve the household's dynamic programming problems (19) satisfy

$$x_j = \frac{1+O(1/x_0)}{\left(\frac{N+1-j}{N+1}\right)\frac{1}{x_0}+\left(\frac{j}{N+1}\right)\frac{1}{x_{N+1}}}$$

Proof. This proof simply combines the previous lemma with the calculations beginning at (34) with the two period nonlinear error replaced by $\epsilon^{(t)}$ in the T period model. □

We conclude by establishing the conditions under which the savings function has non-monotonic second derivative. Once again let $x := x_1$ denote initial resources in this setting, and let $\epsilon(x) := \epsilon^{(1)}(x)$ denote the nonlinear term in the corresponding savings function $k(x) := k^{(1)}(x)$.

Theorem 2: Non-Monotonic Second Derivative With Two Periods. The savings functions $k^{(1)}$ which solve the household's dynamic programming problems (19) with $T = 2$ has a non-monotonic second derivative if and only if

$$\frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)^2} < \frac{1}{1+\beta} \tag{42}$$

Proof. The second derivative of the household savings function is that of the nonlinear term. Recall that we have

$$\epsilon(x) = \frac{D}{k(x) + w_{\mathrm{low}} + \frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)}} \tag{43}$$

with

$$D = \frac{\mathrm{Var}_1(n_2)}{1+\beta} \tag{44}$$

Differentiating, we get

$$\begin{aligned} \frac{\partial \epsilon}{\partial x}(x) &= \frac{-D}{\left(k(x) + w_{\mathrm{low}} + \frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^2} \left(A + \frac{\partial \epsilon}{\partial x}(x)\right) \\ &= \frac{-AD}{\left(k(x) + w_{\mathrm{low}} + \frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^2 + D} \end{aligned} \tag{45}$$

Taking another derivative, we have

$$\begin{aligned} \frac{\partial^2 \epsilon}{\partial x^2}(x) = &\frac{2D}{\left(k(x) + w_{\mathrm{low}} + \frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^3} \left(A + \frac{\partial \epsilon}{\partial x}(x)\right)^2 \\ &- \frac{D}{\left(k(x) + w_{\mathrm{low}} + \frac{\mathrm{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^2} \frac{\partial^2 \epsilon}{\partial x^2}(x) \end{aligned}$$

Solving for the second derivative in this, we obtain the expression

$$\frac{2D}{\left(k(x)+w_{\text{low}}+\frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)}\left(\frac{1}{D+\left(k(x)+w_{\text{low}}+\frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^2}\right)\left(A+\frac{\partial\epsilon}{\partial x}(x)\right)^2$$

and using the expression for the first derivative gives

$$2A^2D\left(\frac{k(x)+w_{\text{low}}+\frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)}}{D+\left(k(x)+w_{\text{low}}+\frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^2}\right)^3$$

From this we can take another derivative and see that its sign will be that of

$$D-\left(k(x)+w_{\text{low}}+\frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^2 \tag{46}$$

As $x\to\infty$ we have $k(x)\to\infty$, so that in this limit the third derivative is negative (the curvature of the savings function is decreasing in wealth for wealthy households).

Consider, then, the limit at the borrowing constraint, namely at the natural borrowing limit. We have

$$\lim_{x\to 0} k(x) = -w_{\text{low}} \tag{47}$$

and the so sign of the third derivative of k at this limit is that of

$$\frac{\text{Var}_1(n_2)}{1+\beta}-\left(\frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)}\right)^2 \tag{48}$$

In other words, the third derivative will be positive (and hence the second derivative will be non-monotonic) provided

$$\frac{\text{Var}_1(n_2)}{\mathbb{E}_1(n_2)^2}<\frac{1}{1+\beta} \tag{49}$$

□