KITT Book Equations Derivation

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1 Introduction

This note contains derivations of the key equations in the KITT book. We also include some additional derivations that are not in the book but may help with understanding and interpreting the model. Section 2 looks at the household's problem, section 3 looks at the firms' problems, and section 4 looks at the foreign firm's problem.

2 Households

This section begins by looking at the household's aggregation problem for labour, the household faces similar problems when aggregating other differentiated goods. Subsection 2 describes the household's problem. Subsection 3 goes through the working for the household's wage setting problem, subsection 4 derives the consumption demand equations, subsection 5 derives the Euler equations and subsection 6 looks at the investment decision.

2.1 ***Labour aggregation***

To introduce sticky wages into the model, labour needs to be differentiated. This allows workers to set their wage as a mark up over the marginal rate of substitution between consumption and labour (the wage that would exist in a perfectly competitive labour market). There is a continuum of labour varieties indexed from 0 to 1. Household's supply all varieties of the differentiated labour input which is aggregated according to Dixit-Stiglitz constant elasticity of substitution (CES) aggregation. Aggregate labour is given by

$$L_t = \left(\int_0^1 \ell_{it}^{1-\frac{1}{\epsilon}} \, \mathrm{d}i \right)^{\frac{\epsilon}{\epsilon-1}}, \tag{1}$$

where L_t denotes aggregate labour, ℓ_{it} is the ith variety of labour for $i \in [0,1]$ and $\epsilon > 1$ is the elasticity of substitution between differentiated labour. This elasticity has to be larger than 1 to ensure that the labour varieties are sufficiently substitutable, and the demand curve sufficiently flat to ensure workers' marginal labour revenue is positive.

The household chooses the quantity of each variety of labour to minimise costs subject to the demand for aggregate labour:

$$\min_{\{\ell_{it}\}_0^1} \quad \int_0^1 w_{it}\ell_{it}\,\mathrm{d}i$$

subject to

$$L_t = \left(\int_0^1 \ell_{it}^{1-\frac{1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}},$$

where $W_t L_t = \int_0^1 w_{it}\ell_{it}\,\mathrm{d}i$ is the aggregate wage bill, and w_{it} is the wage paid to the ith variety of labour.

We set up the Lagrangean for the household

$$\mathcal{L}_t = \int_0^1 w_{it}\ell_{it}\,\mathrm{d}i + \Lambda_t^L\left[L_t - \left(\int_0^1 \ell_{it}^{1-\frac{1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}}\right], \tag{2}$$

where Λ_t^L is the Lagrange multiplier, which is the shadow price or the cost of violating the labour supply constraint by an extra unit.

We get the following first order condition for the ith variety of labour

$$\frac{\partial \mathcal{L}_t}{\partial \ell_{it}} = w_{it} - \Lambda_t^L\left(\frac{\epsilon}{\epsilon-1}\right)\left(\int_0^1 \ell_{it}^{1-\frac{1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{1}{\epsilon-1}}\left(1-\frac{1}{\epsilon}\right)\ell_{it}^{-\frac{1}{\epsilon}} = 0, \tag{3}$$

which we can rearrange to get

$$w_{it} = \Lambda_t^L L_t^{\frac{1}{\epsilon}}\ell_{it}^{-\frac{1}{\epsilon}}, \tag{4}$$

where $L_t^{\frac{1}{\epsilon}} = \left(\int_0^1 \ell_{it}^{1-\frac{1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{1}{\epsilon-1}}$.

Rearranging for ℓ_{it} gives the CES demand for the ith variety of labour

$$\ell_{it} = \left(\frac{w_{it}}{W_t}\right)^{-\epsilon} L_t, \tag{5}$$

where $W_t = \Lambda_t^L$, follows from Λ_t^L being the shadow price, the cost of violating the constraint by increasing aggregate labour by an extra unit. We can interpret Λ_t^L as the marginal cost of increasing labour. Because the aggregation process is perfectly competitive, price equals marginal cost. In this case, price is the aggregate wage rate, W_t.

To find the explicit functional form for the aggregate wage W_t, we can substitute the demand functions for each variety of labour ℓ_{it}, into the CES labour aggregator function, and then solve for the aggregate wage

$$L_t = \left(\int_0^1 \left(\left(\frac{w_{it}}{W_t}\right)^{-\epsilon} L_t\right)^{1-\frac{1}{\epsilon}} \mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}}. \tag{6}$$

We can bring L_t, aggregate labour, to the front of the integral because it is a constant and independent of the variety of labour

$$L_t = L_t \left(\int_0^1 \left(\left(\frac{w_{it}}{W_t}\right)^{-\epsilon}\right)^{1-\frac{1}{\epsilon}} \mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}}. \tag{7}$$

Dividing both sides by L_t gives

$$1 = \left(\int_0^1 \left(\left(\frac{w_{it}}{W_t}\right)^{-\epsilon}\right)^{1-\frac{1}{\epsilon}} \mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}}. \tag{8}$$

Raising both sides to the power $1 - \frac{1}{\epsilon}$ gives

$$1 = \int_0^1 \left(\frac{w_{it}}{W_t}\right)^{1-\epsilon} \mathrm{d}i. \tag{9}$$

Multiplying through by $W_t^{1-\epsilon}$ (because it is independent of labour variety) gives

$$W_t^{1-\epsilon} = \int_0^1 w_{it}^{1-\epsilon}\, \mathrm{d}i. \tag{10}$$

Raising both sides by the power $\frac{1}{1-\epsilon}$ gives

$$W_t = \left[\int_0^1 w_{it}^{1-\epsilon} \, \mathrm{d}i \right]^{\frac{1}{1-\epsilon}}, \tag{11}$$

which is the aggregate wage rate.

2.2 The household's problem

Households derive utility from the consumption of tradable goods, non-tradable goods, petrol and housing services, and disutility from working. Households are infinitely lived, and want to maximise the present value of the sum of their future stream of utilities subject to certain resource constraints.

The household's utility function is given by:

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[U\left(C_t^{\tau}, C_t^{f}, C_t^{n}, C_t^{h} \right) - V\left(\int_0^1 \ell_{it} \, \mathrm{d}i \right) \right]$$

where

$$\begin{aligned} U(\cdots) \equiv\ & \omega_\tau (1-\chi) \log(C_t^{\tau} - \chi \bar{C}_{t-1}^{\tau}) + \omega_f (1-\chi) \log(C_t^{f} - \chi \bar{C}_{t-1}^{f}) \\ & + (1 - \omega_\tau - \omega_f - \omega_h)(1-\chi) \log(C_t^{n} - \exp(\varepsilon_t^{cn}) \, \chi \bar{C}_{t-1}^{n}) + \omega_h \log C_t^{h}, \end{aligned}$$

and

$$V(\cdots) \equiv \frac{1}{1+\eta} \left[\int_0^1 \ell_{it} \, \mathrm{d}i \right]^{1+\eta}.$$

E_0 is the expectations operator conditional on time 0 information, β is the time preference parameter, C_t^{τ} is consumption of tradables, C_t^{n} is consumption of non-tradables, ε_t^{cn} is a shock to non-tradables consumption, C_t^{f} is consumption of petrol, C_t^{h} is consumption of housing services, χ is the weight on deep habit, ω_τ is tradable's share of consumption, ω_f is petrol's share of consumption, ω_h is housing service's share of consumption and η is the inverse of the Frisch elasticity of labour supply. A bar over a variable indicates the variable is an aggregate variable and that the household is too small to influence this directly, so does not take into account its own impact on these variables when making decisions.

Households maximise utility subject to four constraints:

- A resource constraint:

$$
\begin{aligned}
P_t^\tau(C_t^\tau + I_t^k) + P_t^f C_t^f + P_t^n C_t^n + P_t^c I_t^h - B_t \exp(\varepsilon_t^c) = -B_{t-1}(1+r_{t-1}^h) \\
+ \textstyle\int_0^1 w_{it}\ell_{it}\,\mathrm{d}i + R_t K_{t-1} + \Pi_t \\
- \tfrac{1}{2}\phi_c P_t^f \bar{C}_t^f \left(\log C_t^f - \log C_{t-1}^f\right)^2 \\
- \tfrac{1}{2}\iota_h P_t^c \bar{I}_t^h \left(\log I_t^h - \log \bar{I}_{t-1}^h - \varepsilon_t^{ih}\right)^2 \\
- \tfrac{1}{2}\iota_k P_t^\tau \bar{I}_t^k \left(\log I_t^k - \log \bar{I}_{t-1}^k - \varepsilon_t^{ik}\right)^2 \\
- \textstyle\int_0^1 \left[\tfrac{1}{2}\xi_w \bar{W}_t \bar{L}_t \left(\Delta \log w_{it} - \Delta \log \bar{W}_{t-1} - \varepsilon_t^w\right)^2\right] \mathrm{d}i.
\end{aligned}
$$

- The law of motion for business capital:

$$K_t = (1-\delta_k)K_{t-1} + I_t^k.$$

- The law of motion for housing capital:

$$H_t = (1-\delta_h)H_{t-1} + \left(I_t^h\right)^{\gamma_h}.$$

- And the demand for differentiated labour services:

$$\ell_{it} = (w_{it}/W_t)^{-\epsilon}\; L_t.$$

P_t^τ denotes the price of tradables, P_t^f is the price of petrol, P_t^n is the price of non-tradables, P_t^c is the cost of construction, I_t^k is business investment, I_t^h is residential investment, B_t is foreign debt denominated in the local currency, ε_t^c is a consumption preference shock,[1] r_t^h is the effective interest rate, R_t is the rental on business capital, K_t is the stock of business capital, Π_t are profits, ϕ_c is the weight on adjustment costs in the consumption of petrol, ι_h is the weight on the installation costs for residential investment, ι_k is the weight on the installation costs for business investment, ξ_w is the weight on wage adjustment costs, ε_t^w is a wage cost push shock, H_t is the stock of housing capital, and γ_h is residential investment's share in the production of new housing.[2]

2.3 ***Wage setting***

Households maximise utility by setting wages subject to the budget constraint which includes a convex wage adjustment cost. In the presence of a shock,

[1] Note that Smets and Wouters (2007) refer to this as a risk premia shock.
[2] This assumes there is a fixed factor, namely land, in the production of new houses.

these adjustment costs prevent the household from adjusting wages to the flexible price income maximising level, that is wages exhibit some stickiness. Households are able to set their wage rate because labour is differentiated.[3] When households set wages they also take into account how their wage setting decision for each variety of labour impacts the demand for that variety of labour. This means substituting $(w_{it}/W_t)^{-\epsilon}\, L_t$ for every ℓ_{it} in the household's problem.

The disutility of labour in the utility function is given by

$$V(\cdots) \equiv \tfrac{1}{1+\eta}\left[\int_0^1 \ell_{it}\,\mathrm{d}i\right]^{1+\eta}.$$

Substituting in the CES labour demand function for ℓ_{it} (equation (5)) gives:

$$V(\cdots) \equiv \tfrac{1}{1+\eta}\left[\int_0^1 \left((w_{it}/W_t)^{-\epsilon}\, L_t\right)\,\mathrm{d}i\right]^{1+\eta},$$

$$V(\cdots) \equiv \tfrac{1}{1+\eta}\left[W_t^{\epsilon} L_t \left(\int_0^1 w_{it}^{-\epsilon}\,\mathrm{d}i\right)\right]^{1+\eta}.$$

Substituting the CES labour demand function for ℓ_{it} in the labour income term in the budget constraint gives

$$W_t^{\epsilon} L_t \left(\int_0^1 w_{it}^{1-\epsilon}\,\mathrm{d}i\right) = \int_0^1 w_{it}\ell_{it}\,\mathrm{d}i.$$

Substituting these terms into the household's problem and setting up the Lagrangean:

[3] This allows households to set different wages for each variety of labour and not lose their entire market share if wages are set above the aggregate wage, and not gain the entire market share if wages are lower than the aggregate wage.

$$
\mathcal{L}_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ U\left(C_t^{\tau}, C_t^{f}, C_t^{n}, C_t^{h}\right) - \tfrac{1}{1+\eta}\left[W_t^{\epsilon} L_t \left(\int_0^1 w_{it}^{-\epsilon}\,\mathrm{d}i\right)\right]^{1+\eta} \right\}
$$

$$
- \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \left\{ \begin{array}{c} P_t^{\tau}(C_t^{\tau} + I_t^{k}) + P_t^{f} C_t^{f} + P_t^{n} C_t^{n} + P_t^{c} I_t^{h} \\ -B_t \exp(\varepsilon_t^{c}) + B_{t-1}(1 + r_{t-1}^{h}) \\ -W_t^{\epsilon} L_t \left(\int_0^1 w_{it}^{1-\epsilon}\,\mathrm{d}i\right) - R_t K_{t-1} - \Pi_t \\ +\tfrac{1}{2}\phi_c P_t^{f} \bar{C}_t^{f}\left(\log C_t^{f} - \log \bar{C}_{t-1}^{f}\right)^2 \\ +\tfrac{1}{2}\iota_h P_t^{c} \bar{I}_t^{h}\left(\log I_t^{h} - \log \bar{I}_{t-1}^{h} - \varepsilon_t^{ih}\right)^2 \\ +\tfrac{1}{2}\iota_k P_t^{\tau} \bar{I}_t^{k}\left(\log I_t^{k} - \log \bar{I}_{t-1}^{k} - \varepsilon_t^{ik}\right)^2 \\ +\int_0^1 \left[\tfrac{1}{2}\xi_w \bar{W}_t \bar{L}_t \left(\Delta \log w_{it} - \Delta \log \bar{W}_{t-1} - \varepsilon_t^{w}\right)^2\right]\mathrm{d}i \\ +\Phi_t^{k}\left[K_t - (1-\delta_k)K_{t-1} - I_t^{k}\right] \\ +\Phi_t^{h}\exp\left(-\varepsilon_t^{\Phi h}\right)\left[H_t - (1-\delta_h)H_{t-1} - \left(I_t^{h}\right)^{\gamma_h}\right] \end{array} \right\}, \qquad (12)
$$

where Λ_t is the Lagrange multiplier for the budget constraint, and is the shadow value of wealth, the cost of violating the budget constraint by an extra unit. Φ_t^k and Φ_t^h are the Lagrange multipliers associated with the law of motion for business and residential capital respectively. These Lagrange multipliers can be interpreted as the cost of violating the capital constraints, or the marginal cost of capital. $\varepsilon_t^{\Phi h}$ is a shock to house prices.

The first order condition for wages of the ith variety of labour is given by

$$
\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial w_{it}} = \left[W_t^{\epsilon} L_t \left(\int_0^1 w_{it}^{-\epsilon}\,\mathrm{d}i\right)\right]^{\eta} \cdot \epsilon w_{it}^{-\epsilon-1} W_t^{\epsilon} L_t + \Lambda_t (1-\epsilon) w_{it}^{-\epsilon} W_t^{\epsilon} L_t \\
- \Lambda_t \xi_w \tfrac{\bar{W}_t \bar{L}_t}{w_{it}}\left[\Delta \log w_{it} - \Delta \log \bar{W}_{t-1} - \varepsilon_t^{w}\right] \\
+ \mathrm{E}_t\, \beta \Lambda_{t+1} \tfrac{\xi_w \bar{W}_{t+1} \bar{L}_{t+1}}{w_{it}}\left[\Delta \log w_{it+1} - \Delta \log \bar{W}_t\right] = 0. \qquad (13)
\end{aligned}
$$

The steps that follow involve rearranging this first order condition to get back the Phillips curve for wage inflation.

We now allow $\bar{W}_t = W_t$ and $\bar{L}_t = L_t$, which gives

$$
\begin{aligned}
\epsilon \left[\int_0^1 \ell_{it}\,\mathrm{d}i\right]^{\eta} \tfrac{\ell_{it}}{w_{it}} + \Lambda_t (1-\epsilon)\ell_{it} = \Lambda_t \xi_w \tfrac{W_t L_t}{w_{it}}\left[\Delta \log w_{it} - \Delta \log W_{t-1} - \varepsilon_t^{w}\right] \\
- \mathrm{E}_t\, \beta \Lambda_{t+1} \tfrac{\xi_w W_{t+1} L_{t+1}}{w_{it}}\left[\Delta \log w_{it+1} - \Delta \log W_t\right]. \qquad (14)
\end{aligned}
$$

Multiplying through by w_{it} gives

$$\epsilon\left[\int_0^1 \ell_{it}\,\mathrm{d}i\right]^\eta \ell_{it} + \Lambda_t\left(1-\epsilon\right)\ell_{it}w_{it} = \Lambda_t\xi_w W_t L_t\left[\Delta\log w_{it} - \Delta\log W_{t-1} - \varepsilon_t^w\right] \\ - \mathrm{E}_t\,\beta\Lambda_{t+1}\xi_w W_{t+1}L_{t+1}\left[\Delta\log w_{it+1} - \Delta\log W_t\right]. \quad (15)$$

Dividing through by ℓ_{it} gives

$$\epsilon\left[\int_0^1 \ell_{it}\,\mathrm{d}i\right]^\eta + \Lambda_t\left(1-\epsilon\right)w_{it} = \Lambda_t\xi_w \tfrac{W_t L_t}{\ell_{it}}\left[\Delta\log w_{it} - \Delta\log W_{t-1} - \varepsilon_t^w\right] \\ - \mathrm{E}_t\,\beta\Lambda_{t+1}\xi_w \tfrac{W_{t+1}L_{t+1}}{\ell_{it}}\left[\Delta\log w_{it+1} - \Delta\log W_t\right]. \quad (16)$$

We make use of the fact that $\frac{W_t L_t}{\ell_{it}} = \frac{W_t L_t}{(w_{it}/W_t)^{-\epsilon}L_t} = W_t^{1-\epsilon}w_{it}^{\epsilon}$ where we have substituted $(w_{it}/W_t)^{-\epsilon}L_t$ for ℓ_{it}, to get

$$\epsilon\left[\int_0^1 \ell_{it}\,\mathrm{d}i\right]^\eta + \Lambda_t\left(1-\epsilon\right)w_{it} = \Lambda_t\xi_w W_t^{1-\epsilon}w_{it}^{\epsilon}\left[\Delta\log w_{it} - \Delta\log W_{t-1} - \varepsilon_t^w\right] \\ - \mathrm{E}_t\,\beta\Lambda_{t+1}\xi_w \tfrac{W_{t+1}L_{t+1}}{W_t L_t} W_t^{1-\epsilon}w_{it}^{\epsilon}\left[\Delta\log w_{it+1} - \Delta\log W_t\right]. \quad (17)$$

Dividing through by Λ_t

$$\frac{\epsilon\left[\int_0^1 \ell_{it}\,\mathrm{d}i\right]^\eta}{\Lambda_t} + \left(1-\epsilon\right)w_{it} = \xi_w W_t^{1-\epsilon}w_{it}^{\epsilon}\left[\Delta\log w_{it} - \Delta\log W_{t-1} - \varepsilon_t^w\right] \\ - \mathrm{E}_t\,\beta\xi_w \tfrac{\Lambda_{t+1}}{\Lambda_t}\tfrac{W_{t+1}L_{t+1}}{W_t L_t} W_t^{1-\epsilon}w_{it}^{\epsilon}\left[\Delta\log w_{it+1} - \Delta\log W_t\right]. \quad (18)$$

Dividing through by W_t

$$\frac{\epsilon\left[\int_0^1 \ell_{it}\,\mathrm{d}i\right]^\eta}{W_t\Lambda_t} + \left(1-\epsilon\right)\frac{w_{it}}{W_t} = \xi_w\left(\tfrac{w_{it}}{W_t}\right)^{\epsilon}\left[\Delta\log w_{it} - \Delta\log W_{t-1} - \varepsilon_t^w\right] \\ - \mathrm{E}_t\,\beta\xi_w \tfrac{\Lambda_{t+1}}{\Lambda_t}\tfrac{W_{t+1}L_{t+1}}{W_t L_t}\left(\tfrac{w_{it}}{W_t}\right)^{\epsilon}\left[\Delta\log w_{it+1} - \Delta\log W_t\right]. \quad (19)$$

Replacing $(1-\epsilon)$ with $-(\epsilon-1)$ gives

$$\frac{\epsilon\left[\int_0^1 \ell_{it}\,\mathrm{d}i\right]^\eta}{W_t\Lambda_t} - (\epsilon-1)\frac{w_{it}}{W_t} = \xi_w\left(\frac{w_{it}}{W_t}\right)^\epsilon\left[\Delta\log w_{it} - \Delta\log W_{t-1} - \varepsilon_t^w\right]$$
$$- \mathrm{E}_t\,\beta\xi_w\frac{\Lambda_{t+1}}{\Lambda_t}\frac{W_{t+1}L_{t+1}}{W_tL_t}\left(\frac{w_{it}}{W_t}\right)^\epsilon\left[\Delta\log w_{it+1} - \Delta\log W_t\right]. \quad (20)$$

Dividing through by $\epsilon - 1$ gives

$$\frac{\epsilon}{\epsilon-1}\frac{\left[\int_0^1 \ell_{it}\,\mathrm{d}i\right]^\eta}{W_t\Lambda_t} - \frac{w_{it}}{W_t} = \frac{\xi_w}{\epsilon-1}\left(\frac{w_{it}}{W_t}\right)^\epsilon\left[\Delta\log w_{it} - \Delta\log W_{t-1} - \varepsilon_t^w\right]$$
$$- \mathrm{E}_t\,\beta\frac{\xi_w}{\epsilon-1}\frac{\Lambda_{t+1}}{\Lambda_t}\frac{W_{t+1}L_{t+1}}{W_tL_t}\left(\frac{w_{it}}{W_t}\right)^\epsilon\left[\Delta\log w_{it+1} - \Delta\log W_t\right]. \quad (21)$$

In equilibrium $L_t = \int_0^1 \ell_{it}\,\mathrm{d}i$ because $\ell_{jt} = \ell_{it}\;\forall\, i,j \in [0,1]$ so that $\int_0^1 \ell_{it}\,\mathrm{d}i = \left(\int_0^1 \ell_{it}^{1-\frac{1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}}$.

We substitute $\Phi_t^w \equiv \frac{L_t^\eta}{\Lambda_t}$ where the left hand side is the standard first order condition for labour when labour is homogenous, that is the marginal rate of substitution between aggregate labour and consumption in the utility function. This gives

$$\frac{\epsilon}{\epsilon-1}\Phi_t^w/W_t - (w_{it}/W_t) =$$
$$\frac{\xi_w}{\epsilon-1}\left[(w_{it}/W_t)^\epsilon\left(\Delta\log w_{it} - \Delta\log W_{t-1} - \varepsilon_t^w\right) -\right.$$
$$\left.\mathrm{E}_t\,\frac{\beta\Lambda_{t+1}}{\Lambda_t}\frac{W_{t+1}L_{t+1}}{W_tL_t}\left(\frac{w_{it}}{W_t}\right)^\epsilon\left(\Delta\log w_{it+1} - \Delta\log W_t\right)\right]. \quad (22)$$

In a symmetric equilibrium $w_{it} = w_{jt}\;\forall\, i,j \in [0,1]$ so that $W_t = w_{it}\;\forall\, i \in [0,1]$ and

$$\frac{\epsilon}{\epsilon-1}\Phi_t^w/W_t - 1 = \frac{\xi_w}{\epsilon-1}\left[\left(\Delta\log w_{it} - \Delta\log W_{t-1} - \varepsilon_t^w\right) -\right.$$
$$\left.\mathrm{E}_t\,\frac{\beta\Lambda_{t+1}}{\Lambda_t}\frac{W_{t+1}L_{t+1}}{W_tL_t}\left(\Delta\log w_{it+1} - \Delta\log W_t\right)\right]. \quad (23)$$

We define

$$D_t \equiv \frac{\beta\Lambda_{t+1}}{\Lambda_t}\frac{W_{t+1}L_{t+1}}{W_tL_t}\left(\Delta\log w_{it+1} - \Delta\log W_t\right), \quad (24)$$

which is the second term on the right hand side in equation (23). This term is highly non-linear, so we linearise it by taking a first order Taylor series approximation around the steady state.

The general form of a first order Taylor series approximation for a multivariate function around its steady state is

$$f(x_t, y_t, \ldots) \approx f(x_{ss}, y_{ss}, \ldots) + f_{x_t}(x_{ss}, y_{ss}, \ldots)(x_t - x_{ss}) + f_y(x_{ss}, y_{ss}, \ldots)(y_t - y_{ss}) + \ldots$$

Applying this to D_t gives:

$$\begin{aligned}
D_t(\Lambda_{t+1}, \Lambda_t, W_{t+1}, W_t, L_{t+1}, L_t, \Delta \log w_{it+1}, \Delta \log W_t) \approx \\
D(\Lambda, W, L, \Delta \log w_i, \Delta \log W) \\
+ D_{\Lambda_{t+1}}(\Lambda, W, L, \Delta \log w_i, \Delta \log W)(\Lambda_{t+1} - \Lambda) \\
+ D_{\Lambda_t}(\Lambda, W, L, \Delta \log w_i, \Delta \log W)(\Lambda_t - \Lambda) \\
+ D_{W_{t+1}}(\Lambda, W, L, \Delta \log w_i, \Delta \log W)(W_{t+1} - W) \\
+ D_{W_t}(\Lambda, W, L, \Delta \log w_i, \Delta \log W)(W_t - W) \\
+ D_{L_{t+1}}(\Lambda, W, L, \Delta \log w_i, \Delta \log W)(L_{t+1} - L) \\
+ D_{L_t}(\Lambda, W, L, \Delta \log w_i, \Delta \log W)(L_t - L) \\
+ D_{\Delta \log w_{it+1}}(\Lambda, W, L, \Delta \log w_i, \Delta \log W)(\Delta \log w_{it+1} - \Delta \log w_i) \\
+ D_{W_t}(\Lambda, W, L, \Delta \log w_i, \Delta \log W)(\Delta \log W_t - \Delta \log W).
\end{aligned}$$

Evaluating each of these derivatives

$$\frac{\partial D_t}{\partial \Lambda_{t+1}} = \frac{\beta}{\Lambda_t} \frac{W_{t+1}L_{t+1}}{W_t L_t} \left(\Delta \log w_{it+1} - \Delta \log W_t\right),$$

$$\frac{\partial D_t}{\partial \Lambda_t} = -\frac{\beta \Lambda_{t+1}}{\Lambda_t^2} \frac{W_{t+1}L_{t+1}}{W_t L_t} \left(\Delta \log w_{it+1} - \Delta \log W_t\right),$$

$$\frac{\partial D_t}{\partial W_{t+1}} = \frac{\beta \Lambda_{t+1}}{\Lambda_t} \frac{L_{t+1}}{W_t L_t} \left(\Delta \log w_{it+1} - \Delta \log W_t\right),$$

$$\frac{\partial D_t}{\partial W_t} = -\frac{\beta \Lambda_{t+1}}{\Lambda_t} \frac{W_{t+1}L_{t+1}}{W_t^2 L_t} \left(\Delta \log w_{it+1} - \Delta \log W_t\right),$$

$$\frac{\partial D_t}{\partial L_{t+1}} = \frac{\beta \Lambda_{t+1}}{\Lambda_t} \frac{W_{t+1}}{W_t L_t} \left(\Delta \log w_{it+1} - \Delta \log W_t\right),$$

$$\frac{\partial D_t}{\partial L_t} = -\frac{\beta \Lambda_{t+1}}{\Lambda_t} \frac{W_{t+1}L_{t+1}}{W_t L_t^2} \left(\Delta \log w_{it+1} - \Delta \log W_t\right),$$

$$\frac{\partial D_t}{\partial \Delta \log w_{it+1}} = \frac{\beta \Lambda_{t+1}}{\Lambda_t} \frac{W_{t+1}L_{t+1}}{W_t L_t},$$

$$\frac{\partial D_t}{\partial \Delta \log W_t} = -\frac{\beta \Lambda_{t+1}}{\Lambda_t} \frac{W_{t+1}L_{t+1}}{W_t L_t}.$$

Plugging these into the first order Taylor series approximation for D_t gives

$$\begin{aligned}
D_t \approx & \left(\frac{\beta\Lambda}{\Lambda}\frac{WL}{WL}\right)\left(\Delta \log w_i - \Delta \log W\right) \\
& + \left(\frac{\beta}{\Lambda}\frac{WL}{WL}\right)\left(\Delta \log w_i - \Delta \log W\right)\left(\Lambda_{t+1} - \Lambda\right) \\
& - \left(\frac{\beta\Lambda}{\Lambda^2}\frac{WL}{WL}\right)\left(\Delta \log w_i - \Delta \log W\right)\left(\Lambda_t - \Lambda\right) \\
& + \left(\frac{\beta\Lambda}{\Lambda}\frac{L}{WL}\right)\left(\Delta \log w_i - \Delta \log W\right)\left(W_{t+1} - W\right) \\
& - \left(\frac{\beta\Lambda}{\Lambda}\frac{WL}{W^2L}\right)\left(\Delta \log w_i - \Delta \log W\right)\left(W_t - W\right) \\
& + \left(\frac{\beta\Lambda}{\Lambda}\frac{W}{WL}\right)\left(\Delta \log w_i - \Delta \log W\right)\left(L_{t+1} - L\right) \\
& - \left(\frac{\beta\Lambda}{\Lambda}\frac{WL}{WL^2}\right)\left(\Delta \log w_i - \Delta \log W\right)\left(L_t - L\right) \\
& + \left(\frac{\beta\Lambda}{\Lambda}\frac{WL}{WL}\right)\left(\Delta \log w_{it+1} - \Delta \log w_i\right) \\
& - \left(\frac{\beta\Lambda}{\Lambda}\frac{WL}{WL}\right)\left(\Delta \log W_t - \Delta \log W\right).
\end{aligned}$$

Because $\Delta \log w_i = \Delta \log W$ in the steady state, all the terms in D_t disappear except for the last two, so that D_t becomes

$$\begin{aligned}
D_t & \approx \beta\left(\Delta \log w_{it+1} - \Delta \log w_i\right) - \beta\left(\Delta \log W_t - \Delta \log W\right) \\
& \approx \beta\left(\Delta \log w_{it+1} - \Delta \log W_t\right),
\end{aligned}$$

or alternatively we can use the first order Taylor series approximation of an exponential function: $\exp(\hat{x}_t) \approx 1 + \hat{x}_t$ to get

$$\begin{aligned} D_t &\equiv \tfrac{\beta \Lambda_{t+1}}{\Lambda_t} \tfrac{W_{t+1} L_{t+1}}{W_t L_t} \left(\Delta \log w_{it+1} - \Delta \log W_t\right) \\ &\approx \beta(1 + \hat{\lambda}_{t+1} + \hat{w}_{t+1} + \hat{l}_{t+1} - \hat{\lambda}_t - \hat{w}_t - \hat{l}_t)(\pi^w_{t+1} - \pi^w_t) \\ &\approx \beta(\pi^w_{t+1} - \pi^w_t), \end{aligned}$$

because $(\hat{\lambda}_{t+1} + \hat{w}_{t+1} + \hat{l}_{t+1} - \hat{\lambda}_t - \hat{w}_t - \hat{l}_t)(\pi^w_{t+1} - \pi^w_t)$ is very small close to the steady state.

Substituting in the first order approximation of D_t into equation (23) gives the Phillips curve for wages:

$$\pi^w_t - \pi^w_{t-1} = \tfrac{\epsilon - 1}{\xi_w} \left[\left(\tfrac{\epsilon}{\epsilon - 1}\right) \Phi^w_t / W_t - 1 \right] + \beta \, \mathrm{E}_t \left(\pi^w_{t+1} - \pi^w_t \right) + \varepsilon^w_t. \tag{25}$$

We can go one step further and take a first order Taylor series approximation of the real marginal rate of substitution:

$$\begin{aligned} \left(\tfrac{\epsilon}{\epsilon-1}\right) \Phi^w_t / W_t - 1 &= \left(\tfrac{\epsilon}{\epsilon-1}\right) \tfrac{\Phi^w \exp(\hat{\phi}^w_t)}{W \exp(\hat{w}_t)} - 1] \\ &= \left(\tfrac{\epsilon}{\epsilon-1}\right) \tfrac{\Phi^w}{W} \exp(\hat{\phi}^w_t - \hat{w}_t) - 1 \\ &= \exp(\hat{\phi}^w_t - \hat{w}_t) - 1 \\ &\approx 1 + \hat{\phi}^w_t - \hat{w}_t - 1 \\ &\approx \hat{\phi}^w_t - \hat{w}_t, \end{aligned} \tag{26}$$
$$\tag{27}$$

where $\hat{w}_t \equiv \log W_t - \log W$ and $\hat{\phi}^w_t \equiv \log \Phi^w_t - \log \Phi^w$. We make use of the fact that: $\left(\tfrac{\epsilon}{\epsilon-1}\right) \tfrac{\Phi^w}{W} = 1$, in the steady state, and the first order Taylor series approximation for an exponential: $\exp(x_t) \approx 1 + x_t$ for x_t small.

It follows from the definition of the marginal rate of substitution, $\Phi^w_t \equiv \frac{L^\eta_t}{\Lambda_t}$, that

$$\hat{\phi}^w_t = \eta \hat{l}_t - \hat{\lambda}_t, \tag{28}$$

where $\hat{l}_t \equiv \log L_t - \log L$ and $\hat{\lambda}_t \equiv \log \Lambda_t - \log \Lambda$.

This allows us to write the wage Phillips curve as

$$\pi_t^w - \pi_{t-1}^w = \left(\tfrac{\epsilon-1}{\xi_w}\right)\left[\eta\hat{l}_t - \hat{\lambda}_t - \hat{w}_t\right] + \beta\,\mathrm{E}_t\left(\pi_{t+1}^w - \pi_t^w\right) + \varepsilon_t^w. \tag{29}$$

The log deviation of the shadow value for wealth $\hat{\lambda}_t$, will be negatively related to nominal consumption and includes the income effect.

2.4 Consumption

In addition to choosing labour ℓ_{it}, and wages w_{it}, household's choose consumption of tradables C_t^τ, non-tradables C_t^n, and petrol C_t^f, business investment I_t^k, residential investment I_t^h, business capital K_t, housing capital H_t, and debt B_t to maximise the expected sum of discounted future utility.

Setting up the Lagrangean:

$$\mathcal{L}_0 = \mathrm{E}_0\sum_{t=0}^{\infty}\beta^t\left\{\begin{array}{c}\omega_\tau(1-\chi)\log(C_t^\tau - \chi\bar{C}_{t-1}^\tau)\\ +\omega_f(1-\chi)\log(C_t^f - \chi\bar{C}_{t-1}^f)\\ +(1-\omega_\tau-\omega_f-\omega_h)(1-\chi)\log(C_t^n - \exp(\varepsilon_t^{cn})\chi\bar{C}_{t-1}^n)\\ +\omega_h\log H_{t-1} + \omega_h\varepsilon_t^{ch}\\ -V\left(\int_0^1 \ell_{it}\,\mathrm{d}i\right)\end{array}\right\}$$
$$-\,\mathrm{E}_0\sum_{t=0}^{\infty}\beta^t\Lambda_t\left\{\begin{array}{c}P_t^\tau(C_t^\tau + I_t^k) + P_t^f C_t^f + P_t^n C_t^n + P_t^c I_t^h - B_t\exp(\varepsilon_t^c)\\ +B_{t-1}(1+r_{t-1}^h) - \int_0^1 w_{it}\ell_{it}\,\mathrm{d}i - R_t K_{t-1} - \Pi_t\\ +\frac{1}{2}\phi_c P_t^f\bar{C}_t^f\left(\log C_t^f - \log C_{t-1}^f\right)^2\\ +\frac{1}{2}\iota_h P_t^c\bar{I}_t^h\left(\log I_t^h - \log\bar{I}_{t-1}^h - \varepsilon_t^{ih}\right)^2\\ +\frac{1}{2}\iota_k P_t^\tau\bar{I}_t^k\left(\log I_t^k - \log\bar{I}_{t-1}^k - \varepsilon_t^{ik}\right)^2\\ +\int_0^1\left[\frac{1}{2}\xi_w\bar{W}_t\bar{L}_t\left(\Delta\log w_{it} - \Delta\log\bar{W}_{t-1} - \varepsilon_t^w\right)^2\right]\mathrm{d}i\\ +\Phi_t^k\left[K_t - (1-\delta_k)K_{t-1} - I_t^k\right]\\ +\Phi_t^h\exp(-\varepsilon_t^{\Phi h})\left[H_t - (1-\delta_h)H_{t-1} - \left(I_t^h\right)^{\gamma_h}\right]\end{array}\right\}, \tag{30}$$

where we have substituted $C_t^h = H_{t-1}\exp(\varepsilon_t^{ch})$.

The first order conditions for tradable, non-tradable and petrol consumption are given by:

$$\frac{\partial \mathcal{L}_t}{\partial C_t^\tau} = \frac{\omega_\tau(1-\chi)}{C_t^\tau - \chi \bar{C}_{t-1}^\tau} - \Lambda_t P_t^\tau = 0, \tag{31}$$

$$\frac{\partial \mathcal{L}_t}{\partial C_t^n} = \frac{(1-\omega_\tau-\omega_f-\omega_h)(1-\chi)}{C_t^n - \exp(\epsilon_t^{cn})\chi \bar{C}_{t-1}^n} - \Lambda_t P_t^n = 0, \tag{32}$$

$$\frac{\partial \mathcal{L}_t}{\partial C_t^f} = \frac{\omega_f(1-\chi)}{C_t^f - \chi \bar{C}_{t-1}^f} - \Lambda_t P_t^f - \Lambda_t \phi_c P_t^f \bar{C}_t^f \left[\log C_t^f - \log C_{t-1}^f\right]\left(\tfrac{1}{C_t^f}\right) \tag{33}$$

$$+\beta \Lambda_{t+1}\phi_c \frac{P_{t+1}^f C_{t+1}^f}{C_t^f}\left[\log C_{t+1}^f - \log C_t^f\right] = 0, \tag{34}$$

$$\tag{35}$$

which can be rewritten as:[4]

$$C_t^n = \frac{(1-\omega_\tau-\omega_f-\omega_h)(1-\chi)}{P_t^n \Lambda_t} + \exp(\varepsilon_t^{cn})\chi C_{t-1}^n,$$

$$C_t^\tau = \frac{\omega_\tau(1-\chi)}{P_t^\tau \Lambda_t} + \chi C_{t-1}^\tau,$$

$$C_t^f = \frac{\omega_f(1-\chi)}{P_t^f \Lambda_t \varphi_t^C} + \chi C_{t-1}^f,$$

where $\varphi_t^C = 1 + \phi_c\left[\log C_t^f - \log C_{t-1}^f\right] - \beta\phi_c \mathrm{E}_t\left[\log C_{t+1}^f - \log C_t^f\right]$. This adjustment cost term comes from a first order Taylor series approximation of the last term in equation (34). This is derived below. We define[5]

$$D_t(\Lambda_{t+1}, P_{t+1}^f, C_{t+1}^f, C_t^f, \log C_t^f, \log C_{t+1}^f) \equiv \beta\Lambda_{t+1}\phi_c \frac{P_{t+1}^f C_{t+1}^f}{C_t^f}\left[\log C_{t+1}^f - \log C_t^f\right].$$

Taking a first order Taylor series approximation gives:

[4] Dropping the bars because we are in a symmetric equilibrium.

[5] I have ignored the expectations operator to simplify the notation.

$$
\begin{aligned}
D_t \approx\ & D(\Lambda, P^f, C^f, C^f, \log C^f, \log C^f) \\
& + D_{\Lambda_{t+1}}(\Lambda, P^f, C^f, C^f, \log C^f, \log C^f)(\Lambda_{t+1} - \Lambda) \\
& + D_{P^f_{t+1}}(\Lambda, P^f, C^f, C^f, \log C^f, \log C^f)(P^f_{t+1} - P^f) \\
& + D_{C^f_{t+1}}(\Lambda, P^f, C^f, C^f, \log C^f, \log C^f)(C^f_{t+1} - C^f) \\
& + D_{C^f_t}(\Lambda, P^f, C^f, C^f, \log C^f, \log C^f)(C^f_t - C^f) \\
& + D_{\log C^f_t}(\Lambda, P^f, C^f, C^f, \log C^f, \log C^f)(\log C^f_t - \log C^f) \\
& + D_{\log C^f_{t+1}}(\Lambda, P^f, C^f, C^f, \log C^f, \log C^f)(\log C^f_{t+1} - \log C^f).
\end{aligned}
$$

Evaluating the Taylor series approximation around the steady state gives

$$
\begin{aligned}
D_t \approx\ & D_{\log C^f_t}(\Lambda, P^f, C^f, C^f, \log C^f, \log C^f)(\log C^f_t - \log C^f), \\
& + D_{\log C^f_{t+1}}(\Lambda, P^f, C^f, C^f, \log C^f, \log C^f)(\log C^f_{t+1} - \log C^f).
\end{aligned}
$$

This is because $\log C^f_{t+1} = \log C^f_t$ in the steady state, eliminating all other terms.

Evaluating the derivatives gives:

$$
\begin{aligned}
\frac{\partial D_t}{\partial \log C^f_t} &= -\beta \Lambda_{t+1} \phi_c \frac{P^f_{t+1} C^f_{t+1}}{C^f_t} \\
\frac{\partial D_t}{\partial \log C^f_{t+1}} &= \beta \Lambda_{t+1} \phi_c \frac{P^f_{t+1} C^f_{t+1}}{C^f_t}
\end{aligned}
$$

Plugging these back into the Taylor series approximation gives

$$
\begin{aligned}
D_t &\approx \beta \Lambda \phi_c P^f (\log C_{t+1} - \log C) - \beta \Lambda \phi_c P^f (\log C_t - \log C) \\
&\approx \beta \Lambda \phi_c P^f (\log C_{t+1} - \log C).
\end{aligned}
$$

Or alternatively we can make use of the first order Taylor series approximation of the exponential function to get:

$$
\begin{aligned}
\beta\phi_c\Lambda_{t+1}\frac{P^f_{t+1}C^f_{t+1}}{C^f_t}\left[\log C^f_{t+1}-\log C^f_t\right] &= \beta\phi_c\Lambda P^f\exp(\hat{\lambda}_{t+1}+\hat{p}^f_{t+1}+\hat{c}^f_{t+1}-\hat{c}^f_t)\left[\hat{c}^f_{t+1}-\hat{c}^f_t\right] \\
&\approx \beta\phi_c\Lambda P^f\left[1+\hat{\lambda}_{t+1}+\hat{p}^f_{t+1}+\hat{c}^f_{t+1}-\hat{c}^f_t\right]\left[\hat{c}^f_{t+1}-\hat{c}^f_t\right] \\
&\approx \beta\phi_c\Lambda P^f\left[\hat{c}^f_{t+1}-\hat{c}^f_t\right] \\
&\approx \beta\phi_c\Lambda P^f\left[\log C^f_{t+1}-\log C^f_t\right].
\end{aligned}
$$

Where the third line follows from the fact that $\left[\hat{\lambda}_{t+1}+\hat{p}^f_{t+1}+\hat{c}^f_{t+1}-\hat{c}^f_t\right]\left[\hat{c}^f_{t+1}-\hat{c}^f_t\right]$ is negligible.

This approximation does not depend on the time subscripts for P^f and Λ, so that the first order Taylor series approximation of

$$
\beta\Lambda_{t+1}\phi_c\frac{P^f_{t+1}C^f_{t+1}}{C^f_t}\left[\log C^f_{t+1}-\log C^f_t\right] \approx \beta\Lambda_t\phi_c P^f_t\left[\log C^f_{t+1}-\log C^f_t\right].
$$

This allows us to write the first order condition for the consumption of petrol as:

$$
\begin{aligned}
&\Lambda_t P^f_t - \Lambda_t\phi_c P^f_t\bar{C}^f_t\left[\log C^f_t-\log C^f_{t-1}\right]\left(\frac{1}{C^f_t}\right) \\
&\qquad - \beta\Lambda_{t+1}\phi_c\frac{P^f_{t+1}C^f_{t+1}}{C^f_t}\left[\log C^f_{t+1}-\log C^f_t\right] = \\
&\qquad\quad - \Lambda_t P^f_t\Big(1+\phi_c\left[\log C^f_t-\log C^f_{t-1}\right] \\
&\qquad\qquad + \beta\phi_c\left[\log C^f_{t+1}-\log C^f_t\right]\Big).
\end{aligned}
$$

Which we can solve for the demand for petrol.

2.5 Capital and debt: the Euler equations

Household's choose the level of housing capital, business capital and the level of debt to hold. We get the following first order conditions from the Lagrangean:

$$\frac{\partial \mathcal{L}_t}{\partial H_t} = \frac{\beta \omega_h}{H_t} - \Lambda_t \Phi_t^h \exp(-\varepsilon_t^{\Phi h}) + \beta \operatorname{E}_t \Lambda_{t+1} \Phi_{t+1}^h (1-\delta_h) = 0, \tag{36}$$

$$\frac{\partial \mathcal{L}_t}{\partial K_t} = \Lambda_t \Phi_t^k - \beta \operatorname{E}_t \Lambda_{t+1} (R_{t+1} + (1-\delta_k)\Phi_{t+1}^k) = 0, \tag{37}$$

$$\frac{\partial \mathcal{L}_t}{\partial B_t} = -\Lambda_t \exp(\varepsilon_t^c) + \operatorname{E}_t \left(\beta \Lambda_{t+1} (1 + r_t^h)\right) = 0. \tag{38}$$

$$\tag{39}$$

Rearranging these gives us the Euler equations for housing and business capital, and debt, respectively:

$$\Lambda_t \Phi_t^h = \left[\frac{\beta \omega_h}{H_t} + \beta \operatorname{E}_t \Lambda_{t+1} \Phi_{t+1}^h (1-\delta_h)\right] \exp(\varepsilon_t^{\Phi h}), \tag{40}$$

$$\Lambda_t \Phi_t^k = \beta \operatorname{E}_t \Lambda_{t+1} (R_{t+1} + (1-\delta_k)\Phi_{t+1}^k), \tag{41}$$

$$\Lambda_t = \beta \operatorname{E}_t \Lambda_{t+1} (1 + r_t^h) \exp(-\varepsilon_t^c). \tag{42}$$

$$\tag{43}$$

2.5.1 Business capital

We can combine the Euler equation for business capital with the Euler equation for debt and rewrite the result in a more familiar form. This can either take the form of the Fisher equation, or we can express the price of capital goods as an asset price, equal to the sum of future discounted stream of earnings.

Fisher equation form:

Substituting in our debt Euler equation for Λ_t on the left side gives

$$\beta \operatorname{E}_t \Lambda_{t+1} (1 + r_t^h) \exp(-\varepsilon_t^c) \Phi_t^k = \beta \operatorname{E}_t \Lambda_{t+1} (R_{t+1} + (1-\delta_k)\Phi_{t+1}^k). \tag{44}$$

Canceling terms

$$(1 + r_t^h) \exp(-\varepsilon_t^c) = \operatorname{E}_t \left[\frac{R_{t+1}}{\Phi_t^k} + (1-\delta_k)\frac{\Phi_{t+1}^k}{\Phi_t^k}\right]. \tag{45}$$

Taking a first order Taylor series approximation of the shock term on the left side ($\exp(-\varepsilon_t^c) \approx 1 - \varepsilon_t^c$), and rewriting the gross inflation rate of business investment prices in net terms, gives

$$(1+r_t^h)(1-\varepsilon_t^c) \approx \mathrm{E}_t\left[\tfrac{R_{t+1}}{\Phi_t^k} + (1-\delta_k)(1+\tfrac{\dot{\Phi}_{t+1}^k}{\Phi_t^k})\right], \tag{46}$$

where $\Phi_{t+1}^k = \dot{\Phi}_{t+1}^k + \Phi_t^k$.

Expanding these terms gives

$$1 + r_t^h - \varepsilon_t^c - r_t^h\varepsilon_t^c \approx \mathrm{E}_t\left[\tfrac{R_{t+1}}{\Phi_t^k} + 1 - \delta_k + \tfrac{\dot{\Phi}_{t+1}^k}{\Phi_t^k} - \delta_k\tfrac{\dot{\Phi}_{t+1}^k}{\Phi_t^k}\right]. \tag{47}$$

We can rewrite this as

$$1 + r_t^h - \varepsilon_t^c \approx \mathrm{E}_t\left[\tfrac{R_{t+1}}{\Phi_t^k} + 1 - \delta_k + \tfrac{\dot{\Phi}_{t+1}^k}{\Phi_t^k}\right], \tag{48}$$

because $r_t^h\varepsilon_t^c$ and $\delta_k\frac{\dot{\Phi}_{t+1}^k}{\Phi_t^k}$ are small.

We can then rearrange this into its Fisher equation form:

$$\mathrm{E}_t\left(\tfrac{R_{t+1}}{\Phi_t^k} - \delta_k\right) \approx r_t^h - \mathrm{E}_t\left(\tfrac{\dot{\Phi}_{t+1}^k}{\Phi_t^k}\right) - \varepsilon_t^c, \tag{49}$$

$$\mathrm{E}_t\left((1-\gamma_{z1}-\gamma_{z2})\left(\tfrac{P_{t+1}^z Z_{t+1}}{\Phi_t^k K_t}\right) - \delta_k\right) \approx r_t^h - \mathrm{E}_t\left(\tfrac{\dot{\Phi}_{t+1}^k}{\Phi_t^k}\right) - \varepsilon_t^c, \tag{50}$$

where the term on the left side is the real return on business capital, and the term on the right side is the nominal effective interest rate less the inflation rate on the price of business investment (i.e. the real interest rate for this sector).[6]

Asset price form:

We can also rewrite the price of business investment in its asset price form, that is equal to its expected future discounted stream of earnings or marginal products. From equation (44), we have

$$\beta\,\mathrm{E}_t\,\Lambda_{t+1}(1+r_t^h)\exp(-\varepsilon_t^c)\Phi_t^k = \beta\,\mathrm{E}_t\,\Lambda_{t+1}(R_{t+1} + (1-\delta_k)\Phi_{t+1}^k). \tag{51}$$

Dividing both sides by $\beta\,\mathrm{E}_t\,\Lambda_{t+1}(1+r_t^h)\exp(-\varepsilon_t^c)$, gives

[6] Note that we use the equilibrium condition $K_t' = K_{t-1}$.

$$\Phi_t^k = \frac{\mathrm{E}_t(R_{t+1} + (1-\delta_k)\Phi_{t+1}^k)\exp(\varepsilon_t^c)}{1+r_t^h}. \tag{52}$$

We can then substitute in all future values of Φ_t^k to obtain:[7]

$$\Phi_0^k = \mathrm{E}_0 \sum_{t=0}^{\infty} \left[\frac{R_{t+1}(1-\delta_k)^t}{\Pi_{j=0}^t (1+r_j^h)} \right] \exp(\varepsilon_0^c), \tag{53}$$

$$\Phi_0^k = \mathrm{E}_0 \sum_{t=0}^{\infty} \left[\frac{(1-\gamma_{z1}-\gamma_{z2})\left(\frac{P_{t+1}^z Z_{t+1}}{K_t}\right)(1-\delta_k)^t}{\Pi_{j=0}^t (1+r_j^h)} \right] \exp(\varepsilon_0^c). \tag{54}$$

The price of a new business investment good today is equal to the expected discounted sum of earnings (marginal products) from the quantity of investment goods purchased today, where the marginal product at date $t+1$ of a unit of capital bought at date 0 is equal to $R_{t+1}(1-\delta_k)^t$. We can see this if we substitute date 0 investment into the production function for intermediate goods and differentiate with respect to date 0 investment at some future date $t+1$.

The production function for intermediate goods at date $t+1$ is given by[8]

$$Z_{t+1} = (F_{t+1}^z)^{\gamma_{z1}} \left[A_{t+1}(L_{t+1}-L_0)\right]^{\gamma_{z2}} (K_t)^{1-\gamma_{z1}-\gamma_{z2}},$$

where Z_{t+1} is the intermediate good, F_{t+1}^z is oil, A_{t+1} is technology, L_{t+1} is labour, L_0 is overhead labour, K_t is business capital, γ_{z1} is oil's share of production and γ_{z2} is labour's share of production.

We can write the capital stock as the sum of past investment

$$\begin{aligned} K_t &= I_t^k + (1-\delta_k)K_{t-1} \\ &= \sum_{j=1}^{t} I_j^k (1-\delta_k)^{t-j} + I_0^k(1-\delta_k)^t + (1-\delta_k)^{t+1}K_{-1}. \end{aligned}$$

Plugging this into the date $t+1$ production function gives

[7] Using the equilibrium condition $K_t' = K_{t-1}$.
[8] Using the equilibrium condition $K_t' = K_{t-1}$.

$$Z_{t+1} = (F^z_{t+1})^{\gamma_{z1}} \; [A_{t+1}(L_{t+1} - L_0)]^{\gamma_{z2}} \Bigg(\sum_{j=1}^{t} I^k_j (1-\delta_k)^{t-j} + $$
$$I^k_0 (1-\delta_k)^t + (1-\delta_k)^{t+1} K_{-1} \Bigg)^{1-\gamma_{z1}-\gamma_{z2}} .$$

Differentiating with respect to date 0 business investment gives

$$\begin{aligned} \frac{\partial (P^z_{t+1} Z_{t+1})}{\partial I^k_0} &= (1-\gamma_{z1}-\gamma_{z2}) P^z_{t+1} (F^z_{t+1})^{\gamma_{z1}} \; [A_{t+1}(L_{t+1} - L_0)]^{\gamma_{z2}} \times \\ & \left(\sum_{j=1}^{t} I^k_j (1-\delta_k)^{t-j} + I^k_0 (1-\delta_k)^t + (1-\delta_k)^{t+1} K_{-1} \right)^{-\gamma_{z1}-\gamma_{z2}} (1-\delta_k)^t \\ &= (1-\gamma_{z1}-\gamma_{z2}) P^z_{t+1} (F^z_{t+1})^{\gamma_{z1}} \; [A_{t+1}(L_{t+1} - L_0)]^{\gamma_{z2}} \, (K_t)^{-\gamma_{z1}-\gamma_{z2}} (1-\delta_k)^t \\ &= (1-\gamma_{z1}-\gamma_{z2}) P^z_{t+1} \frac{Z_{t+1}}{K_t} (1-\delta_k)^t \\ &= R_{t+1} (1-\delta_k)^t \end{aligned}$$

Which shows that $R_{t+1}(1-\delta)^t$ is the marginal product of date 0 business investment at date $t+1$.

2.5.2 Housing capital

We can also combine the Euler equation for housing capital and the Euler equation for debt and then rewrite this result, either in its Fisher equation form or as an asset price.

Fisher equation form:

Combining the Euler equation for housing with the Euler equation for debt gives:

$$\Lambda_t \Phi^h_t \exp(-\varepsilon^{\Phi h}_t) = \left[\frac{\beta \omega_h}{H_t} + \beta \, \mathrm{E}_t \, \Lambda_{t+1} \Phi^h_{t+1} (1-\delta_h) \right], \tag{55}$$

$$\beta \, \mathrm{E}_t \, \Lambda_{t+1} (1+r^h_t) \exp(-\varepsilon^c_t - \varepsilon^{\Phi h}_t) \Phi^h_t = \left[\frac{\beta \omega_h}{H_t} + \beta \, \mathrm{E}_t \, \Lambda_{t+1} \Phi^h_{t+1} (1-\delta_h) \right]. \tag{56}$$

Dividing through by $\beta \, \mathrm{E}_t \, \Lambda_{t+1}$ gives

$$(1+r_t^h)\exp(-\varepsilon_t^c - \varepsilon_t^{\Phi h}) = \mathrm{E}_t\left[\frac{\omega_h}{\Phi_t^h \Lambda_{t+1} H_t} + \left(\frac{\Phi_{t+1}^h}{\Phi_t^h}\right)(1-\delta_h)\right]. \tag{57}$$

Taking a first order Taylor series approximation of the shock terms ($\exp(-\varepsilon_t^c - \varepsilon_t^{\Phi h}) \approx 1 - \varepsilon_t^c - \varepsilon_t^{\Phi h}$) and rewriting the gross inflation rate of house prices gives:

$$(1+r_t^h)(1-\varepsilon_t^c - \varepsilon_t^{\Phi h}) \approx \mathrm{E}_t\left[\frac{\omega_h}{\Phi_t^h \Lambda_{t+1} H_t} + \left(1 + \frac{\dot{\Phi}_{t+1}^h}{\Phi_t^h}\right)(1-\delta_h)\right], \tag{58}$$

where $\Phi_{t+1}^h = \dot{\Phi}_{t+1}^h + \Phi_t^h$.

Expanding out the terms gives

$$1+r_t^h - \varepsilon_t^c - \varepsilon_t^{\Phi h} - r_t^h \varepsilon_t^c - r_t^h \varepsilon_t^{\Phi h} \approx \mathrm{E}_t\left[\frac{\omega_h}{\Phi_t^h \Lambda_{t+1} H_t} + 1 + \frac{\dot{\Phi}_{t+1}^h}{\Phi_t^h} - \delta_h - \delta_h \frac{\dot{\Phi}_{t+1}^h}{\Phi_t^h}\right] \tag{59}$$

Which simplifies to

$$r_t^h - \varepsilon_t^c - \varepsilon_t^{\Phi h} \approx \mathrm{E}_t\left[\frac{\omega_h}{\Phi_t^h \Lambda_{t+1} H_t} + \frac{\dot{\Phi}_{t+1}^h}{\Phi_t^h} - \delta_h\right] \tag{60}$$

because $r_t^h \varepsilon_t^c$, $r_t^h \varepsilon_t^{\Phi h}$ and $\delta_h \frac{\dot{\Phi}_{t+1}^h}{\Phi_t^h}$ are small.

We can then rearrange to get this in Fisher equation form:

$$\mathrm{E}_t\left[\frac{\omega_h}{\Phi_t^h \Lambda_{t+1} H_t} - \delta_h\right] \approx r_t^h - \mathrm{E}_t\left(\frac{\dot{\Phi}_{t+1}^h}{\Phi_t^h}\right) - \varepsilon_t^c - \varepsilon_t^{\Phi h} \tag{61}$$

$$\mathrm{E}_t\left(\frac{P_{t+1}^{hs}}{\Phi_t^h} - \delta_h\right) \approx r_t^h - \mathrm{E}_t\left(\frac{\dot{\Phi}_{t+1}^h}{\Phi_t^h}\right) - \varepsilon_t^c - \varepsilon_t^{\Phi h} \tag{62}$$

where we define $P_{t+1}^{hs} \equiv \frac{\omega_h}{\Lambda_{t+1} H_t}$ as the rental on housing services.

Asset price form:

We can also rewrite the price of housing as an asset price, the sum of the discounted future stream of earnings. We begin from equation (55)

$$\beta \,\mathrm{E}_t \,\Lambda_{t+1}(1+r_t^h)\exp(-\varepsilon_t^c)\Phi_t^h = \left[\frac{\beta\omega_h}{H_t} + \beta \,\mathrm{E}_t \,\Lambda_{t+1}\Phi_{t+1}^h(1-\delta_h)\right]\exp(\varepsilon_t^{\Phi h}). \tag{63}$$

Dividing through by β and $\mathrm{E}_t \,\Lambda_{t+1}$ gives

$$(1+r_t^h)\exp(-\varepsilon_t^c)\Phi_t^h = \mathrm{E}_t\left[\frac{\omega_h}{\Lambda_{t+1}H_t} + \Phi_{t+1}^h(1-\delta_h)\right]\exp(\varepsilon_t^{\Phi h}). \tag{64}$$

Dividing through by $(1+r_t^h)\exp(-\varepsilon_t^c)$ gives

$$\Phi_t^h = \mathrm{E}_t\left[\frac{\frac{\omega_h}{\Lambda_{t+1}H_t} + \Phi_{t+1}^h(1-\delta_h)}{1+r_t^h}\right]\exp(\varepsilon_t^c + \varepsilon_t^{\Phi h}). \tag{65}$$

Solving this forward by substituting in the terms for future house prices gives

$$\Phi_0^h = \mathrm{E}_0\sum_{t=0}^{\infty}\left[\frac{\frac{\omega_h}{\Lambda_{t+1}H_t}(1-\delta_h)^t}{\Pi_{j=0}^t(1+r_j^h)}\right]\exp(\varepsilon_0^c + \varepsilon_0^{\Phi h}), \tag{66}$$

$$\Phi_0^h = \mathrm{E}_0\sum_{t=0}^{\infty}\left[\frac{P_{t+1}^{hs}(1-\delta_h)^t}{\Pi_{j=0}^t(1+r_j^h)}\right]\exp(\varepsilon_0^c + \varepsilon_0^{\Phi h}). \tag{67}$$

We can see that house prices can be expressed as the sum of their discounted future stream of (imputed) earnings or marginal utilities of housing services.

2.5.3 Consumption euler equations

Combining the first order condition for debt with the first order conditions for non-tradables consumption, tradables consumption and consumption of petrol gives their respective Euler equations

$$\mathrm{E}_t\left(\frac{C_{t+1}^n - \chi C_t^n}{C_t^n - \exp(\varepsilon_t^{cn})\chi C_{t-1}^n}\right) = \beta \,\mathrm{E}_t\left(\frac{1+r_t^h}{1+\pi_{t+1}^n}\right)\exp(-\varepsilon_t^c), \tag{68}$$

$$\mathrm{E}_t\left(\frac{C_{t+1}^\tau - \chi C_t^\tau}{C_t^\tau - \chi C_{t-1}^\tau}\right) = \beta \,\mathrm{E}_t\left(\frac{1+r_t^h}{1+\pi_{t+1}^\tau}\right)\exp(-\varepsilon_t^c), \tag{69}$$

$$\mathrm{E}_t\left(\frac{C_{t+1}^f - \chi C_t^f}{C_t^f - \chi C_{t-1}^f}\right) = \beta \,\mathrm{E}_t\left(\frac{1+r_t^h}{1+\pi_{t+1}^f}\right)\left(\frac{\varphi_t^C}{\varphi_{t+1}^C}\right)\exp(-\varepsilon_t^c). \tag{70}$$

$$\tag{71}$$

2.6 Investment

Households also choose housing investment and business investment, which gives us the following first order conditions

$$\frac{\partial \mathcal{L}_t}{\partial I_t^k} = -\Lambda_t P_t^\tau - \Lambda_t \iota_k P_t^\tau \bar{I}_t^k \left[\log I_t^k - \log \bar{I}_{t-1}^k - \varepsilon_t^{ik}\right] \left(\frac{1}{I_t^k}\right) + \Lambda_t \Phi_t^k = 0, \quad (72)$$

$$\frac{\partial \mathcal{L}_t}{\partial I_t^h} = -\Lambda_t P_t^c - \Lambda_t \iota_h P_t^c \bar{I}_t^h \left[\log I_t^h - \log \bar{I}_{t-1}^h - \varepsilon_t^{ih}\right] \left(\frac{1}{I_t^h}\right) + \Lambda_t \Phi_t^H \gamma_h (I_t^h)^{\gamma_h - 1} = 0. \quad (73)$$

In symmetric equilibrium $\bar{I}_t^h = I_t^h$ and $\bar{I}_t^k = I_t^k$ so that

$$\Phi_t^k / P_t^\tau = 1 + \iota_k \left(\log I_t^k - \log I_{t-1}^k - \varepsilon_t^{ik}\right), \quad (74)$$

$$\frac{\gamma_h \Phi_t^h \left(I_t^h\right)^{\gamma_h - 1}}{P_t^c} = 1 + \iota_h \left(\log I_t^h - \log I_{t-1}^h - \varepsilon_t^{ih}\right). \quad (75)$$

These can be interpreted as the demand functions for business and residential investment respectively. They are also the supply functions for new business capital and new housing capital, respectively.

Installation costs on business and residential investment imply demand for business and residential capital is downward sloping in the short run, and the supply of new business and housing capital is upward sloping in the short run. To see this we can log linearise equations (74) and (75), and rearrange to get

$$\hat{i}_t^k = \left(\frac{1}{\iota_k}\right) \left[\hat{\phi}_t^k - \hat{p}_t^\tau\right] + \hat{i}_{t-1}^k + \varepsilon_t^{ik}, \quad (76)$$

$$\hat{i}_t^h = \left(\frac{1}{\iota_h - \gamma_h + 1}\right) \left[\hat{\phi}_t^h - \hat{p}_t^c + \iota_h (\hat{i}_{t-1}^h + \varepsilon_t^{ih})\right], \quad (77)$$

where $\hat{i}_t^k \equiv \log\left(\frac{I_t^k}{I^k}\right)$, $\hat{\phi}_t^k \equiv \log\left(\frac{\Phi_t^k}{\Phi^k}\right)$ $\hat{p}_t^\tau \equiv \log\left(\frac{P_t^\tau}{P^\tau}\right)$, $\hat{i}_t^h \equiv \log\left(\frac{I_t^h}{I^h}\right)$, $\hat{\phi}_t^h \equiv \log\left(\frac{\Phi_t^h}{\Phi^h}\right)$ and $\hat{p}_t^c \equiv \log\left(\frac{P_t^c}{P^c}\right)$.

Business investment $\hat{i}_t^k$ is negatively related to the price of tradable goods $\hat{p}_t^\tau$, implying that demand for business investment is downward sloping. Business investment is positively related to the price of business capital $\hat{\phi}_t^k$ implying that the supply of new business capital is upward sloping. $-\left(\frac{1}{\iota_k}\right)$ is the own price elasticity of demand, and $\left(\frac{1}{\iota_k}\right)$ and the own price elasticity of supply. Note that in the steady state $\hat{i}_t^k = \hat{i}_{t-1}^k$, implying that supply and demand are perfectly elastic in the steady state.

We have a similar interpretation for residential investment, except there are decreasing returns in the production of new houses, which means landlords face a downward sloping demand curve for residential investment even in the steady state. The supply of new houses is also upward sloping in the steady state.

If we assume that $Y_t^h = (I_t^h)^{\gamma_h}$ is the production function for new housing, and we ignore the adjustment cost terms, we can rewrite the demand for residential investment as

$$I_t^h = \gamma_h \frac{\Phi_t^h Y_t^h}{P_t^c}, \tag{78}$$

which shows that I_t^h is negatively related to P_t^c (demand is downward sloping).

Rearranging for Φ_t^h the price of new housing gives

$$\Phi_t^h = \frac{P_t^c I_t^h}{\gamma_h Y_t^h}. \tag{79}$$

Substituting in $I_t^h = (Y_t^h)^{\frac{1}{\gamma_h}}$ gives

$$\Phi_t^h = \frac{P_t^c (Y_t^h)^{\frac{1}{\gamma_h}-1}}{\gamma_h},$$

where $\frac{1}{\gamma_h} - 1 > 0$, ensures that we have an upward sloping supply curve for new houses.

3 Firms

In this section we solve the firms' problems in the intermediate goods producing sector, the tradables sector, the non-tradables sector, the construction sector and manufactured exports producing sector.

3.1 ***Intermediate goods producing sector***

Intermediate goods producers maximise the sum of their discounted future stream of profits

$$
\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \Big\{ P_t^z Z_t - P_t^f F_t^z - W_t L_t - R_t K_t' \\
- \tfrac{1}{2}\, \phi_z P_t^f \bar{F}_t^z \left[\log(F_t^z/\bar{Z}_t) - \log(\bar{F}_{t-1}^z/\bar{Z}_{t-1})\right]^2 \Big\},
$$

subject to a production function,

$$
Z_t = (F_t^z)^{\gamma_{z1}}\ [A_t(L_t - L_0)]^{\gamma_{z2}}\,(K_t')^{1-\gamma_{z1}-\gamma_{z2}},
$$

by choosing oil, labour and capital in each period.

P_t^z denotes the price of intermediate goods, Z_t is the intermediate good, F_t^z is the quantity of oil used in the production of intermediate goods, ϕ_z is an adjustment cost on the demand for oil, γ_{z1} is oil's share of income, γ_{z2} is labour's share of income, K_t' is the intermediate firm's demand for capital, L_0 is overhead labour, A_t is technology. Bars over variables indicate that the firm is too small to take into account its own actions on these variables directly when optimising.

We set up the Lagrangean:

$$
\mathcal{L}_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \left\{ \begin{array}{c} P_t^z Z_t - P_t^f F_t^z - W_t L_t - R_t K_t' \\ -\frac{1}{2}\, \phi_z P_t^f \bar{F}_t^z \left[\log(F_t^z/\bar{Z}_t) - \log(\bar{F}_{t-1}^z/\bar{Z}_{t-1})\right]^2 \end{array} \right\} \\
- \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \Phi_t^z \left\{ Z_t - (F_t^z)^{\gamma_{z1}}\ [A_t(L_t - L_0)]^{\gamma_{z2}}\,(K_t')^{1-\gamma_{z1}-\gamma_{z2}} \right\}, \quad (80)
$$

where Φ_t^z is the Lagrange multiplier associated with the demand constraint. This can also be interpreted as the marginal cost.

We get the following first order conditions:

$$\frac{\partial \mathcal{L}_t}{\partial K'_t} = -R_t + (1 - \gamma_{z1} - \gamma_{z2})\frac{\Phi^z_t Z_t}{K'_t} = 0, \tag{81}$$

$$\frac{\partial \mathcal{L}_t}{\partial L_t} = -W_t + \gamma_{z2}\frac{\Phi^z_t Z_t}{L_t - L_0} = 0, \tag{82}$$

$$\frac{\partial \mathcal{L}_t}{\partial F^z_t} = -P^f_t + \gamma_{z1}\frac{\Phi^z_t Z_t}{F^z_t} = 0, \tag{83}$$

$$\tag{84}$$

which we can rearrange to get the demand functions for capital, labour and oil, respectively:

$$K'_t = (1 - \gamma_{z1} - \gamma_{z2})\frac{P^z_t Z_t}{R_t}, \tag{85}$$

$$L_t = \gamma_{z2}\frac{P^z_t Z_t}{W_t} + L_0, \tag{86}$$

$$F^z_t = \gamma_{z1}\frac{P^z_t Z_t}{P^f_t}. \tag{87}$$

$$\tag{88}$$

Perfect competition implies that price is set equal to marginal cost, $P^z_t = \Phi^z_t$.

3.2 ***Tradables***

Tradables goods producers maximise the sum of their discounted future stream of profits

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \Big\{ p^\tau_{it} y^\tau_{it} - P^f_t f^\tau_{it} - P^z_t z^\tau_{it} - P^q_t m^q_{it} \exp(\psi^q + \varepsilon^{mq}_t) \\ - \tfrac{1}{2}\,\phi_\tau P^f_t F^\tau_t \left[\log(f^\tau_{it}/Y^\tau_t) - \log(F^\tau_{t-1}/Y^\tau_{t-1})\right]^2 \\ - \tfrac{1}{2}\,\xi_\tau P^\tau_t Y^\tau_t \left[\Delta \log p^\tau_{it} - \Delta \log P^\tau_{t-1} - \varepsilon^{p\tau}_t\right]^2 \Big\},$$

subject to a production function

$$y^\tau_{it} = A^\tau_t (f^\tau_{it})^{\gamma_{\tau 1}} (z^\tau_{it})^{\gamma_{\tau 2}} (m^q_{it} \exp(\psi^q + \varepsilon^{mq}_t))^{1-\gamma_{\tau 1}-\gamma_{\tau 2}},$$

and a CES demand function

$$y^\tau_{it} = (p^\tau_{it}/P^\tau_t)^{-\epsilon} Y^\tau_t,$$

where $Y_t^\tau = \left(\int_0^1 (y_{it}^\tau)^{1-\frac{1}{\epsilon}}\,\mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}}$ is aggregate tradables, y_{it}^τ is firm i's production of tradable goods, $P_{it}^\tau = \left(\int_0^1 (p_{it}^\tau)^{1-\epsilon}\,\mathrm{d}i\right)^{\frac{1}{1-\epsilon}}$ is the aggregate price of tradables, p_{it}^τ is the price of the ith variety of tradable good. f_{it}^τ,z_{it}^τ and m_{it}^q are firm i's demands for oil, the intermediate good and non-oil imports in the production of tradable goods, respectively. F_t^τ is the aggregate demand for oil in the tradable sector, ϕ_τ is the weight on the adjustment costs for oil, ξ_τ is the weight on the price adjustment costs (controls the degree of price stickiness), A_t^τ is a sector specific tradable technology process, ψ^q is a scaling parameter, ε_t^{mq} is a non-oil import shock, $\gamma_{\tau 1}$ is oil's share of production and $\gamma_{\tau 2}$ is intermediate's share of production.

Setting up the Lagrangean:

$$\begin{aligned}\mathcal{L}_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \Big\{ & p_{it}^\tau y_{it}^\tau - P_t^f f_{it}^\tau - P_t^z z_{it}^\tau - P_t^q m_{it}^q \exp(\psi^q + \varepsilon_t^{mq}) \\ & - \tfrac{1}{2}\,\phi_\tau P_t^f F_t^\tau \left[\log(f_{it}^\tau / Y_t^\tau) - \log(F_{t-1}^\tau / Y_{t-1}^\tau)\right]^2 \\ & - \tfrac{1}{2}\,\xi_\tau P_t^\tau Y_t^\tau \left[\Delta \log p_{it}^\tau - \Delta \log P_{t-1}^\tau - \varepsilon_t^{p\tau}\right]^2 \Big\} \\ - \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \Phi_{it}^\tau \Big\{ & y_{it}^\tau - A_t^\tau (f_{it}^\tau)^{\gamma_{\tau 1}} (z_{it}^\tau)^{\gamma_{\tau 2}} (m_{it}^q \exp(\psi^q + \varepsilon_t^{mq}))^{1-\gamma_{\tau 1}-\gamma_{\tau 2}} \Big\}. \end{aligned} \tag{89}$$

Substituting in the CES demand functions for y_{it}^τ

$$\begin{aligned}\mathcal{L}_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \Big\{ & p_{it}^\tau (p_{it}^\tau / P_t^\tau)^{-\epsilon} Y_t^\tau - P_t^f f_{it}^\tau - P_t^z z_{it}^\tau - P_t^q m_{it}^q \exp(\psi^q + \varepsilon_t^{mq}) \\ & - \tfrac{1}{2}\,\phi_\tau P_t^f F_t^\tau \left[\log(f_{it}^\tau / Y_t^\tau) - \log(F_{t-1}^\tau / Y_{t-1}^\tau)\right]^2 \\ & - \tfrac{1}{2}\,\xi_\tau P_t^\tau Y_t^\tau \left[\Delta \log p_{it}^\tau - \Delta \log P_{t-1}^\tau - \varepsilon_t^{p\tau}\right]^2 \Big\} \\ - \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \Phi_{it}^\tau \Big\{ & (p_{it}^\tau / P_t^\tau)^{-\epsilon} Y_t^\tau - A_t^\tau (f_{it}^\tau)^{\gamma_{\tau 1}} (z_{it}^\tau)^{\gamma_{\tau 2}} (m_{it}^q \exp(\psi^q + \varepsilon_t^{mq}))^{1-\gamma_{\tau 1}-\gamma_{\tau 2}} \Big\}. \end{aligned} \tag{90}$$

We get the following first order conditions:

$$\frac{\partial \mathcal{L}_t}{\partial f_{it}^z} = -P_t^f - \tfrac{\phi_\tau P_t^f F_t^\tau}{f_{it}^\tau}\left[\log(f_{it}^\tau/Y_t^\tau) - \log(F_{t-1}^\tau/Y_{t-1}^\tau)\right]$$

$$+\gamma_{\tau 1}\left(\frac{\Phi_{it}^\tau y_{it}^\tau}{f_{it}^\tau}\right) = 0, \tag{91}$$

$$\frac{\partial \mathcal{L}_t}{\partial z_{it}^\tau} = -P_t^z + \gamma_{\tau 2}\left(\frac{\Phi_{it}^\tau y_{it}^\tau}{z_{it}^\tau}\right) = 0, \tag{92}$$

$$\frac{\partial \mathcal{L}_t}{\partial m_{it}^q} = -P_t^q \exp(\psi^q + \varepsilon_t^{mq}) + (1-\gamma_{\tau 1}-\gamma_{\tau 2})\left(\frac{\Phi_{it}^\tau y_{it}^\tau}{m_{it}^q}\right) = 0, \tag{93}$$

$$\tag{94}$$

$$\frac{\partial \mathcal{L}_t}{\partial p_{it}^\tau} = (1-\epsilon)\left(p_{it}^\tau/P_{it}^\tau\right)^{-\epsilon} Y_t^\tau - \xi_\tau P_t^\tau Y_t^\tau\left[\Delta \log p_{it}^\tau - \Delta \log P_{t-1}^\tau - \varepsilon_t^{p\tau}\right]\left(\tfrac{1}{p_{it}^\tau}\right)$$
$$+ \beta \xi_\tau P_{t+1}^\tau Y_{t+1}^\tau\left[\Delta \log p_{it+1}^\tau - \Delta \log P_t^\tau\right]\left(\tfrac{1}{p_{it}^\tau}\right) + \epsilon \Phi_{it}^\tau (p_{it}^\tau)^{-\epsilon-1}(P_t^\tau)^\epsilon Y_t^\tau = 0. \tag{95}$$

Rearranging these, gives the following demand equations:

$$\gamma_{\tau 1}\,\Phi_{it}^\tau y_{it}^\tau = P_t^f f_{it}^\tau + \phi_\tau P_t^f F_t^\tau\left[\log(f_{it}^\tau/Y_t^\tau) - \log(F_{t-1}^\tau/Y_{t-1}^\tau)\right], \tag{96}$$

$$\gamma_{\tau 2}\,\Phi_{it}^\tau y_{it}^\tau = P_t^z\, z_{it}^\tau, \tag{97}$$

$$(1-\gamma_{\tau 1}-\gamma_{\tau 2})\,\Phi_{it}^\tau y_{it}^\tau = P_t^q\, m_{it}^q \exp(\psi^q + \varepsilon_t^{mq}). \tag{98}$$

The first order condition for prices can be rewritten as

$$\tfrac{\epsilon}{\epsilon-1}\,\Phi_{it}^\tau/P_t^\tau - (p_{it}^\tau/P_t^\tau) =$$
$$\tfrac{\xi_\tau}{\epsilon-1}\left[A_\tau\left(\Delta \log p_{it}^\tau - \Delta \log P_{t-1}^\tau - \varepsilon_t^{p\tau}\right) - \mathrm{E}_t B_\tau\left(\Delta \log p_{it+1}^\tau - \Delta \log P_t^\tau\right)\right], \tag{99}$$

where

$$A_\tau \equiv \left(p_{it}^\tau/P_t^\tau\right)^\epsilon, \tag{100}$$

$$B_\tau \equiv \frac{\beta \Lambda_{t+1}}{\Lambda_t}\,\frac{P_{t+1}^\tau Y_{t+1}^\tau}{P_t^\tau\, Y_t^\tau}\left(\frac{p_{it}^\tau}{P_t^\tau}\right)^\epsilon. \tag{101}$$

Substituting equations (96), (97) and (98) into the tradable production function and canceling terms gives

$$\frac{\Phi_t^\tau}{P_t^\tau} = \left(\frac{1}{A_t^\tau P_t^\tau}\right)\left(\frac{P_t^f \varphi_t^f}{\gamma_{\tau 1}}\right)^{\gamma_{\tau 1}}\left(\frac{P_t^z}{\gamma_{\tau 2}}\right)\left(\frac{P_t^q}{1-\gamma_{\tau 1}-\gamma_{\tau 2}}\right)^{1-\gamma_{\tau 1}-\gamma_{\tau 2}}, \tag{102}$$

where $\varphi_t^f = 1+\phi_\tau \left[\log\left(F_t^\tau/Y_t^\tau\right) - \log\left(F_{t-1}^\tau/Y_{t-1}^\tau\right)\right]$ is the first derivative of the oil adjustment cost with respect to the demand for oil. $\frac{\Phi_t^\tau}{P_t^\tau}$ can be interpreted as the real marginal cost in the tradable sector. Note that we can drop the i subscript on the real marginal cost term because their is a common factor market.

We can rearrange equation (99), and take a first order Taylor series expansion to get the Phillips curve for the tradable sector, just as we did for the wage Phillips curve (equation (29)) in the household's problem

$$\pi_t^\tau - \pi_{t-1}^\tau = \left(\frac{\epsilon-1}{\xi_\tau}\right)\left[\gamma_{\tau 1}\hat{p}_t^{f/\tau} - \gamma_{\tau 2}\hat{p}_t^{z/\tau} - (1-\gamma_{\tau 1}-\gamma_{\tau 2})\hat{p}_t^{q/\tau} - \hat{a}_t^\tau\right] + \beta\,\mathrm{E}_t\left(\pi_{t+1}^\tau - \pi_t^\tau\right) + \varepsilon_t^{p\tau}, \tag{103}$$

where $\hat{p}_t^{f/\tau} \equiv \hat{p}_t^f + \phi_\tau\left[\hat{f}_t^\tau - \hat{f}_{t-1}^\tau - \hat{y}_t^\tau + \hat{y}_{t-1}^\tau\right] - \hat{p}_t^\tau$, $\hat{p}_t^{z/\tau} \equiv \hat{p}_t^z - \hat{p}_t^\tau$, $\hat{p}_t^{q/\tau} \equiv \hat{p}_t^q - \hat{p}_t^\tau$ and $\hat{a}^\tau \equiv \log\left(\frac{A_t^\tau}{A^\tau}\right)$.

3.3 ***Non-tradables***

Non-tradables firms maximise the sum of their discounted future stream of profits

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \left\{ p_{it}^n y_{it}^n - P_t^z z_{it}^n - \tfrac{1}{2}\xi_n P_t^n Y_t^n \left[\Delta \log p_{it}^n - \Delta \log P_{t-1}^n - \varepsilon_t^{pn}\right]^2 \right\}, \tag{104}$$

subject to the production function

$$y_{it}^n = A_t^n \left(z_{it}^n\right)^{\gamma_n}, \tag{105}$$

and the CES demand for their variety of product

$$y_{it}^n = \left(p_{it}^n/P_t^n\right)^{-\epsilon} Y_t^n, \tag{106}$$

where p_{it}^n is the price set by the ith firm, y_{it}^n is the demand for the ith firm's variety, z_{it}^n is the ith firm's demand for the intermediate good, P_t^n is the aggregate price of non-tradable goods, Y_t^n is the aggregate non-tradable output, A_t^n is the sector specific technology for non-tradables, γ_n is the intermediate good's share of non-tradables, ξ_n is a sector specific price adjustment cost parameter that controls the stickiness of non-tradables goods prices.

Setting up the Lagrangean

$$\mathcal{L}_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \left\{ p_{it}^n \left(p_{it}^n / P_t^n \right)^{-\epsilon} Y_t^n - P_t^z z_{it}^n \right.$$
$$\left. - \tfrac{1}{2} \xi_n P_t^n Y_t^n \left[\Delta \log p_{it}^n - \Delta \log P_{t-1}^n - \varepsilon_t^{pn} \right]^2 \right\}$$
$$- \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \Phi_{it}^n \left\{ \left(p_{it}^n / P_t^n \right)^{-\epsilon} Y_t^n - A_t^n \left(z_{it}^n \right)^{\gamma_n} , \right\}, \quad (107)$$

where Φ_{it}^n denotes the Lagrange multiplier for the production constraint. The firm chooses the quantity of intermediate goods and the price for their variety of good. We then obtain the following first order condition with respect to intermediate goods.

$$\gamma_n \Phi_{it}^n y_{it}^n = P_t^z z_{it}^n, \quad (108)$$

$$z_{it}^n = \frac{\gamma_n \Phi_{it}^n y_{it}^n}{P_t^z}, \quad (109)$$

and with respect to the ith firm's price p_{it}^n

$$\tfrac{\epsilon}{\epsilon-1} \Phi_{it}^n / P_t^n - \left(p_{it}^n / P_t^n \right) =$$
$$\tfrac{\xi_n}{\epsilon-1} \left[A_n \left(\Delta \log p_{it}^n - \Delta \log P_{t-1}^n - \varepsilon_t^{pn} \right) - \mathrm{E}_t \, B_n \left(\Delta \log p_{it+1}^n - \Delta \log P_t^n \right) \right], \quad (110)$$

where

$$A_n \equiv \left(p_{it}^n / P_t^n \right)^{\epsilon},$$

$$B_n \equiv \frac{\beta\Lambda_{t+1}}{\Lambda_t}\frac{P_{t+1}^n Y_{t+1}^n}{P_t^n Y_t^n}\left(\frac{p_{it}^n}{P_t^n}\right)^{\epsilon},$$

$$\frac{\Phi_{it}^n}{P_t^n} = \frac{P_t^z z_{it}^n}{\gamma_n P_t^n y_{it}^\tau} = (1/\gamma_n)\left(\frac{P_t^z}{P_t^n A_t^\tau}\right)\left(\frac{y_{it}^n}{A_t^n}\right)^{\frac{1}{\gamma_n}-1}.$$

In equilibrium $A_n = 1$ and $B_n = \frac{\beta\Lambda_{t+1}}{\Lambda_t}\frac{P_{t+1}^n Y_{t+1}^n}{P_t^n Y_t^n}$. Taking a first order Taylor series expansion of equation (110) around its steady state (i.e. $\Delta\log p_i^n = \Delta\log P^n = \pi^n$) gives $B_n\left(\Delta\log p_{it+1}^n - \Delta\log P_t^n\right) \approx \beta(\pi_{t+1}^n - \pi_t^n)$. We obtain the following Phillips curve relationship

$$\pi_t^n - \pi_{t-1}^n = \left(\tfrac{\epsilon-1}{\xi_n}\right)\left[\left(\tfrac{\epsilon}{\epsilon-1}\right)(\Phi_t^n/P_t^n) - 1\right] + \mathrm{E}_t\,\beta\left(\pi_{t+1}^n - \pi_t^n\right) + \varepsilon_t^{pn}. \tag{111}$$

We can take a first order Taylor series expansion of the real marginal cost term around its steady state:

$$\begin{aligned}\left(\tfrac{\epsilon}{\epsilon-1}\right)(\Phi_t^n/P_t^n) - 1 &= \left(\tfrac{\epsilon}{\epsilon-1}\right)\tfrac{\Phi^n}{P^n}\tfrac{\exp(\hat{\phi}^n)}{\exp(\hat{p}^n)} - 1\\ &= \exp(\hat{\phi}_t^n - \hat{p}_t^n) - 1\\ &\approx 1 + \hat{\phi}_t^n - \hat{p}_t^n - 1\\ &\approx \hat{\phi}_t^n - \hat{p}_t^n,\end{aligned} \tag{112}$$

where $\hat{\phi}_t^n \equiv \log\left(\frac{\Phi_t^n}{\Phi^n}\right)$ and $\hat{p}_t^n \equiv \log\left(\frac{P_t^n}{P^n}\right)$.

From the definition of marginal cost, $\Phi_t^n = (1/\gamma_n)\left(\frac{P_t^z}{A_t^n}\right)\left(\frac{Y_t^n}{A_t^n}\right)^{\frac{1}{\gamma_n}-1}$:[9]

$$\hat{\phi}_t^n = \hat{p}_t^z - \hat{a}_t^n + \left(\frac{1}{\gamma_n} - 1\right)(\hat{y}_t^n - \hat{a}_t^n), \tag{113}$$

where $\hat{a}_t^n \equiv \log\left(\frac{A_t^n}{A^n}\right)$ and $\hat{y}_t^n \equiv \log\left(\frac{Y_t^n}{Y^n}\right)$.

Substituting this back into equation (111) gives

$$\pi_t^n - \pi_{t-1}^n = \left(\frac{\epsilon-1}{\xi_n}\right)\left[\hat{p}_t^{z/n} - \hat{a}_t^n + \left(\frac{1}{\gamma_n} - 1\right)(\hat{y}_t^n - \hat{a}_t^n)\right] + \mathrm{E}_t\,\beta\left(\pi_{t+1}^n - \pi_t^n\right) + \varepsilon_t^{pn}, \tag{114}$$

where $\hat{p}_t^{z/n} \equiv \hat{p}_t^z - \hat{p}_t^n = \log\left(\frac{P_t^z}{P^z}\right) - \log\left(\frac{P_t^n}{P^n}\right)$.

3.4 ***Construction***

Firms producing residential investment goods, maximise the sum of their discounted future stream of profits

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \left\{ p_{it}^c y_{it}^c - P_t^z z_{it}^c - \tfrac{1}{2}\xi_c P_t^c Y_t^c \left[\Delta \log p_{it}^c - \Delta \log P_{t-1}^c - \varepsilon_t^{pc}\right]^2 \right\},$$

subject to a production function

$$y_{it}^c = A_t^c \left(z_{it}^c\right)^{\gamma_c},$$

[9] Rearranging equation (108) for marginal cost gives

$$\Phi_{it}^n = \frac{P_t^z z_{it}^n}{\gamma_n y_{it}^n}.$$

Rearranging the production function for $z_{it}^n = \left(\frac{y_{it}^n}{A_t^n}\right)^{\frac{1}{\gamma_n}}$ and substituting this into our marginal cost term gives

$$\Phi_{it}^n = \left(\frac{1}{\gamma_n}\right)\left(\frac{P_t^z}{A_t^n}\right)\left(\frac{y_{it}^n}{A_t^n}\right)^{\frac{1}{\gamma_n}-1}.$$

Common factor markets means that in a symmetric equilibrium we can drop the i subscripts.

and the CES demand function for their variety of residential investment good

$$y_{it}^c = (p_{it}^c / P_t^c)^{-\epsilon} \, Y_t^c,$$

where p_{it}^c is the price of the ith firm's output, y_{it}^c is the demand for the ith firm's output, z_{it}^c is the ith construction firm's demand for intermediate goods, P_t^c is the aggregate price for residential investment, Y_t^c is aggregate residential investment, ξ_c is a sector specific adjustment cost, ε_t^{pc} is a construction cost push shock, A_t^c is sector specific technology in the construction sector and γ_c is intermediate's share of production in residential investment.

Denoting the lagrange multiplier associated with the production function as Φ_{it}^c gives the first order condition with respect to the intermediate good,

$$\gamma_c \Phi_{it}^c y_{it}^c = P_t^z z_{it}^c, \tag{115}$$

and the first order condition with respect to the price as,

$$\begin{aligned} &\tfrac{\epsilon}{\epsilon-1} \Phi_{it}^c / P_t^c - (p_{it}^c / P_t^c) = \\ &\quad \tfrac{\xi_c}{\epsilon-1} \left[A_c \left(\Delta \log p_{it}^c - \Delta \log P_{t-1}^c - \varepsilon_t^{pc} \right) - \mathrm{E}_t \, B_c \left(\Delta \log p_{it+1}^c - \Delta \log P_t^c \right) \right], \end{aligned} \tag{116}$$

where

$$A_c \equiv (p_{it}^c / P_t^c)^{\epsilon},$$

$$B_c \equiv \frac{\beta \Lambda_{t+1}}{\Lambda_t} \frac{P_{t+1}^c Y_{t+1}^c}{P_t^c Y_t^c} \left(\frac{p_{it}^c}{P_t^c} \right)^{\epsilon},$$

$$\frac{\Phi_{it}^c}{P_t^c} = \frac{P_t^z z_{it}^c}{\gamma_c P_t^c y_{it}^c} = (1/\gamma_c) \left(\frac{P_t^z}{P_t^c A_t^c} \right) \left(\frac{y_{it}^c}{A_t^c} \right)^{\frac{1}{\gamma_c} - 1}, \tag{117}$$

where $\frac{\Phi_{it}^c}{P_t^c}$ can be interpreted as the real marginal cost faced by the ith firm in the construction sector.

Taking a first order Taylor series approximation of the last term in equation (116) and the real marginal cost term, and with some rearranging, we get the following Phillips curve

$$\pi_t^c - \pi_{t-1}^c = \left(\frac{\epsilon-1}{\xi_c}\right)\left[\hat{p}_t^{z/c} - \hat{a}_t^c + \left(\frac{1}{\gamma_c} - 1\right)(\hat{y}_t^c - \hat{a}_t^c)\right] + \mathrm{E}_t\,\beta(\pi_{t+1}^c - \pi_t^c) + \varepsilon_t^{pc}$$

Where $\hat{p}_t^{z/c} \equiv \hat{p}_t^z - \hat{p}_t^c = \log\left(\frac{P_t^z}{P^z}\right) - \log\left(\frac{P_t^c}{P^c}\right)$, $\hat{y}_t^c \equiv \log\left(\frac{Y_t^c}{Y^c}\right)$ and $\hat{a}_t^c \equiv \log\left(\frac{A_t^c}{A^c}\right)$.

3.5 ***Manufactured exports***

Firms producing manufactured exports producers maximise the sum of their discounted future stream of profits

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{S_t} \left\{ p_{it}^{v*} x_{it}^v - S_t P_t^z z_{it}^v - \frac{1}{2}\xi_v P_t^{v*} X_t^v \left[\Delta \log p_{it}^{v*} - \Delta \log P_{t-1}^{v*} - \varepsilon_t^{pv}\right]^2 \right\},$$

subject to a production function

$$x_{it}^v = A_t^v \left(z_{it}^v\right)^{\gamma_v},$$

and a CES demand function

$$x_{it}^v = \left(p_{it}^{v*}/P_t^{v*}\right)^{-\epsilon} X_t^v,$$

where p_{it}^{v*} is the price set by the ith manufacturing export firm, x_{it}^v is the demand for the ith firm's variety of export good, z_{it}^v is the ith exporting firm's demand for the intermediate good, ξ_v is a sector specific cost parameter, A_t^v is sector specific technology in the manufactured export sector and γ_v is the intermediate's share of manufactured exports.

Letting Φ_{it}^v denote the lagrange multiplier associated with the production constraint, we get the first order condition associated with the intermediate good

$$\gamma_v \Phi_{it}^v x_{it}^v = P_t^v z_{it}^v, \tag{118}$$

and the first order condition associated with prices,

$$\tfrac{\epsilon}{\epsilon-1}(S_t\Phi_{it}^v)/P_t^{v*} - (p_{it}^{v*}/P_t^{v*}) =$$
$$\tfrac{\xi_v}{\epsilon-1}\left[A_v\left(\Delta\log p_{it}^{v*} - \Delta\log P_{t-1}^{v*} - \varepsilon_t^{pv}\right) - \mathrm{E}_t\, B_v\left(\Delta\log p_{it+1}^{v*} - \Delta\log P_t^{v*}\right)\right], \tag{119}$$

where

$$A_v \equiv (p_{it}^{v*}/P_t^{v*})^{\epsilon},$$

$$B_v \equiv \frac{\beta\Lambda_{t+1}}{\Lambda_t}\frac{S_t}{S_{t+1}}\frac{P_{t+1}^{v*}X_{t+1}^v}{P_t^{v*}X_t^v}\left(\frac{p_{it}^{v*}}{P_t^{v*}}\right)^{\epsilon}.$$

$$\frac{\Phi_{it}^v}{P_t^v} = \frac{P_t^z z_{it}^v}{\gamma_v P_t^v x_{it}^v} = (1/\gamma_v)\left(\frac{P_t^z}{P_t^v A_t^v}\right)\left(\frac{x_{it}^v}{A_t^v}\right)^{\frac{1}{\gamma_v}-1}, \tag{120}$$

where $\frac{\Phi_{it}^v}{P_t^v}$ can be interpreted as the real marginal cost (in domestic currency) of producing manufactured exports by firm i. Taking a Taylor series approximation of the second term on the right hand side of equation (119) and the real marginal cost term, and rearranging gives the following Phillips curve:

$$\pi_t^{v*} - \pi_{t-1}^{v*} = \left(\tfrac{\epsilon-1}{\xi_v}\right)\left[\hat{p}^{z/v*} - \hat{a}_t^v + \left(\tfrac{1}{\gamma_v}-1\right)(\hat{x}_t^v - \hat{a}_t^v)\right] + \mathrm{E}_t\,\beta(\pi_{t+1}^{v*} - \pi_t^{v*}) + \varepsilon_t^{pv} \tag{121}$$

Where $\hat{p}^{z/v*} \equiv \hat{p}_t^z - \hat{p}_t^{v*} = \log\left(\frac{P_t^z}{P^z}\right) - \log\left(\frac{P_t^v}{P^v}\right)$, $\hat{a}_t^v \equiv \log\left(\frac{A_t^v}{A^v}\right)$ and $\hat{x}_t^v \equiv \log\left(\frac{X_t^v}{X^v}\right)$.

4 Foreign economy

The foreign economy produces output according to the Cobb Douglas production technology:

$$Y_t^* = \left(X_t^d\right)^{\omega}(X_t^v)^{1-\omega}$$

Where Y_t^* is foreign GDP, X_t^d is the demand for domestically produced commodity exports, X_t^v is the demand for domestically produced manufactured exports and ω is commodity's share of income.

The foreign economy is perfectly competitive, and maximises profits:

$$
\begin{aligned}
\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t^* \Big\{ & P_t^{w*} Y_t^* - P_t^{w*} X_t^d - P_t^{v*} X_t^v - \tfrac{1}{2} \eta_d P_t^{w*} \overline{X}_t^d \left[\log X_t^d - \log \overline{X}_{t-1}^d - \varepsilon_t^{xd} \right]^2 \\
& - \tfrac{1}{2} \eta_v P_t^{v*} \overline{X}_t^v \left[\log X_t^v - \log \overline{X}_{t-1}^v - \varepsilon_t^{xv} \right]^2 \Big\}
\end{aligned}
$$

subject to the production function

$$
Y_t^* = \left(X_t^d \right)^{\omega} \left(X_t^v \right)^{1-\omega},
$$

by choosing the quantities of commodity and manufactured imports. This gives us the following first order conditions:

$$
\omega \frac{P_t^{w*} Y_t^*}{P_t^{d*} X_t^d} = 1 + \eta_d \left[\log X_t^d - \log X_{t-1}^d - \varepsilon_t^{xd} \right], \tag{122}
$$

$$
(1-\omega) \frac{P_t^{w*} Y_t^*}{P_t^{v*} X_t^v} = 1 + \eta_v \left[\log X_t^v - \log X_{t-1}^v - \varepsilon_t^{xv} \right], \tag{123}
$$

where Λ_t^* is the marginal utility of consumption in the foreign country, P_t^{w*} is the world price level denominated in foreign currency, P_t^{v*} is the aggregate price level for manufactured exports denoted in foreign currency, η_d and η_x are the weights on the adjustment costs for commodity and manufactured exporters respectively, and ε_t^{xd} and ε_t^{xv} are demand shocks to commodity and manufactured exporters respectively.

We believe that commodity exports are not price sensitive, that is there is a time to build element to them, whatever is produced is sold on the world market. For this reason we drop the prices from the commodity export demand equation so that equation (122) becomes

$$
\omega \frac{Y_t^*}{X_t^d} = 1 + \eta_d \left[\log X_t^d - \log X_{t-1}^d - \varepsilon_t^{xd} \right]. \tag{124}
$$

References

Smets, F. and R. Wouters (2007). Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach. *American Economic Review 97*(3), 586–606.