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## Sharing a Risky Cake\*

David Baqaee and Richard Watt<sup>†</sup>

### Abstract

Consider an  $n$ -person bargaining problem where players bargain over the division of a cake whose size is stochastic. In such a game, the players are not only bargaining over the division of a cake, but they are also sharing risk. This paper presents the Nash bargaining solution to this problem, investigates its properties, and highlights a few special cases.

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# 1 Introduction

Imagine a game where rational players bargain over a set of outcomes. The payoffs are determined by mutual agreement between the players. In a seminal paper, Nash (1950) devised a simple axiomatic characterization of the solution to this general bargaining problem. Nash's paper has since spawned a vast literature on bargaining theory and its applications.

In this paper, we investigate the case where the set of outcomes is the utility possibility frontier arising from the division of a cake whose size is stochastically determined. Players must agree on how to divide the cake before the cake's size is known. Intuitively, in such a situation, we would expect risk-averse players to forgo larger shares of the cake when the cake is big to insure themselves against the possibility that the cake is small. On the other hand, we would expect less risk-averse players to gamble away small stakes for a chance at having large amounts of cake. This game can serve as an allegory for many "real" bargaining situations, particularly those where an unknown amount of money, brought about through cooperation, is to be divided.

We study the cooperative Nash bargaining solution in such a game and find that it confirms this intuition. We characterise and investigate the properties of the solution for the case where  $n$ -players with different beliefs bargain over a one-dimensional random variable. The contribution of this paper is solving and analysing the properties of the Nash bargaining solution without presupposing that the players hold the same beliefs or that the cake is a discrete random variable.

In the next section, we briefly review the Nash bargaining problem and solution. Then, in section 3, we set out and solve the risky cake-sharing problem. In section 4 we briefly discuss a weakening of the Nash axioms and the extension of our results to the non-symmetric case. Proofs are found in the appendix.

## 2 The Nash bargaining problem

**Definition 2.1.** For each natural number  $n$ , an  $n$ -person Nash bargaining problem is any list  $(S, d)$ , where  $S$  is a nonempty, convex, and compact

subset of  $\mathbb{R}^n$  and  $d$  is a strictly dominated element of  $S$  — that is, there exists at least one  $x \in S$  with each component of  $x$  being greater than each component of  $d$ .

The set  $S$  represents the utility possibility set that arises from agreement between the players and  $d$  denotes the outcome if the players fail to reach an agreement. In this game, the players are assumed to be expected utility maximisers.

**Definition 2.2.** Let  $(S, d)$  be a Nash bargaining problem. The **Nash bargaining solution** is a point  $c(S, d)$  in  $S$  that satisfies the following conditions:

**(PE)** The solution  $c(S, d)$  is **Pareto-efficient**; that is, if some outcome  $x$  is Pareto-superior to  $c(S, d)$ , then  $x \notin S$ .

**(AM)** The solution  $c(S, d)$  is invariant to equivalent representations of utility functions; that is, transforming  $S$  by an increasing affine transformation results in an equivalent transformation of the Nash bargaining solution.

**(IIA)** The solution is independent of irrelevant alternatives; that is, if  $T \subseteq S$  and  $c(S, d) \in T$ , then  $c(T, d) = c(S, d)$ .

**(SYM)** If  $S$  is symmetric and all players face the same disagreement outcome, then  $c(S, d)$  assigns the same utility to all players.

There is an extensive literature on the appropriateness of these axioms (see Muthoo 1999). We simply take them as given in this paper.<sup>1</sup>

The following theorem, originally proved for the 2-person case by Nash (1950), and generalised to the  $n$ -person case by Harsanyi (1997), forms the basis for our results.

**Theorem 2.1** (Nash). *For each  $n$ -person Nash bargaining problem  $(S, d)$ , the Nash bargaining solution  $c(S, d)$  exists, is unique, and*

$$c(S, d) \in \operatorname{argmax}_{V \in S} \prod_{i=1}^n (V_i - d_i), \quad (1)$$

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<sup>1</sup> The results we establish in this paper can easily be adapted to deal with the non-symmetric Nash Bargaining solution whose axioms are weaker. See section 4.

where  $V_i$  is the  $i$ th component of the vector  $\mathbf{V}$  in  $S$ . The product in (1) is called the **Nash product**.

Suppose that a deterministic one-dimensional cake of size  $x$  is shared between players 1 and 2. Denote player  $i$ 's utility function by  $V_i$  and let

$$\tilde{w} \in \operatorname{argmax}_{0 \leq w \leq x} (V_1(w) - d_1)(V_2(x - w) - d_2).$$

Then theorem 2.1 implies that  $(V_1(\tilde{w}), V_2(x - \tilde{w}))$  is the Nash bargaining solution. In this case, the cake is partitioned into  $\tilde{w}$  and  $x - \tilde{w}$ . In the next section, we investigate the stochastic analogue to this problem.

### 3 The Nash bargaining problem with risk

For clarity of exposition and simplicity of notation, we begin with a restricted version of the problem and make generalisations as we progress. In this section, we study the case where 2 players with identical beliefs bargain over a random variable. Then, in section 3.1, we extend the analysis to the case where the two players have differing beliefs. Finally, we turn our attention to the  $n$ -player case with differing beliefs.

To begin with, suppose that players 1 and 2 bargain over the partition of a random cake  $X$  (a random variable) with an associated probability distribution  $F$  (this may correspond to a continuous, discrete or mixed distribution function). Further suppose that  $F$  has support  $\Omega \subset \mathbb{R}$ . The players must devise a contract for sharing the cake conditional on its realisation. The contract  $\tilde{w} : \Omega \rightarrow \mathbb{R}$  corresponding to the Nash bargaining solution must then be the solution to the following optimisation problem:

$$\max_w (\mathbb{E}[u_1(w(x))] - d_1)(\mathbb{E}[u_2(x - w(x))] - d_2), \quad (2)$$

where  $\mathbb{E}[\cdot]$  denotes the expectation operator,  $x$  the total amount of cake and  $w(x)$  the amount of cake allocated to player 1 given  $X = x$ . In this paper, the expectation operator of each player's utility function is always taken with respect to the beliefs of that same player. This is to avoid us having to write out the explicit integral every time or unnecessarily complicating notation with subscripts for each player.

It is important to note that because our results are based on differential calculus, they only apply to cases where there is an interior solution – our first order conditions and comparative static results will break down if contract that maximises the Nash product is in a corner.

Due to axiom (AM), we can assume without loss of generality that the disagreement point  $(d_1, d_2)$  is equal to the origin. For this problem, we can prove the following theorem.

**Theorem 3.1.** *Suppose that  $\tilde{w} : \Omega \rightarrow \mathbb{R}$  solves the Nash bargaining problem described above. Let*

$$\begin{aligned} C_1 &= E[u_1(\tilde{w}(y))] &= \int_{\Omega} u_1(\tilde{w}(y)) \, dF(y) \\ C_2 &= E[u_2(y - \tilde{w}(y))] &= \int_{\Omega} u_2(y - \tilde{w}(y)) \, dF(y) \end{aligned}$$

*If the realisation of the cake is  $x$ , the amount  $\tilde{w} : \Omega \rightarrow \mathbb{R}$  given to player 1 must satisfy*

$$\frac{u'_1(\tilde{w}(x))}{u'_2(x - \tilde{w}(x))} = \frac{C_1}{C_2} \quad (F - \text{almost everywhere}). \quad (3)$$

The term  $u'_i$  denotes the derivative of  $u$  with respect to its argument. Note that the phrase “ $F$ -almost everywhere” means that (3) need only hold on sets of nonzero probability. In other words, if we define the event  $B$  by

$$B := \left\{ x \in \Omega : \frac{u'_1(\tilde{w}(x))}{u'_2(x - \tilde{w}(x))} \neq \frac{E[u_1(\tilde{w}(y))]}{E[u_2(y - \tilde{w}(y))]} \right\},$$

then,  $F(B) = 0$ . It is important to note that the right hand side of (3) is a *constant* in  $x$  — that is, the contract  $\tilde{w}$  which solves the Nash bargaining problem must equate the ratio of the marginal utilities to a constant number for every outcome. This is a first order condition for the optimisation problem.

This theorem is our work-horse, it allows us to explicitly solve for the solution of many numerical examples. It also enables us to investigate the properties of the solution in the general case. Unfortunately, the notation is a little awkward and hard to interpret, so it is helpful to consider a specific example to sharpen our intuition about what the result actually implies.

Before we go on, note that (3) implies that the distribution of the cake does not affect the functional form of the solution (it enters the first order condition as a constant). As we shall see later, this observation fails to hold when the players have differing beliefs.

**Example 3.1.** Consider the case where the utility functions  $u_1$  and  $u_2$  of player 1 and 2 are

$$u_1(x) := \sqrt{x} \text{ and } u_2(x) := \log(x).$$

Note that player 2 is more risk-averse than player 1, because the second utility function is an increasing concave transformation of the first (up to multiplication by a scalar). Suppose further that the cake is continuously distributed and its support is some set  $\Omega$  of non-negative real numbers. By theorem 3.1, the contract path  $\tilde{w}$ , specifying player 1's share of cake, will satisfy

$$\frac{\tilde{w}(x) - x}{2\sqrt{\tilde{w}(x)}} = \frac{\int(\sqrt{\tilde{w}(y)}) dF(y)}{\int(\log(y - \tilde{w}(y))) dF(y)} \quad (\text{almost everywhere}).$$

Now note that the right hand side is a constant in  $x$ . Let

$$\alpha := \frac{\int(\sqrt{\tilde{w}(y)}) dF(y)}{\int(\log(y - \tilde{w}(y))) dF(y)}$$

and solve for  $\tilde{w}$  to get

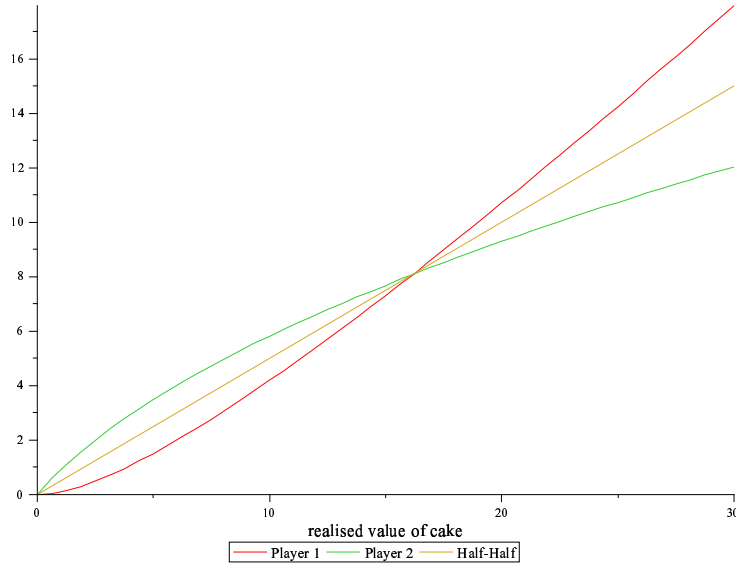
$$\tilde{w}(x) = x - 2\alpha \left( -\alpha + \sqrt{\alpha^2 + x} \right)$$

This pins down the functional form of the contract curve. If we now specify a distribution for the cake, we can numerically solve for the value of the constant  $\alpha$ . For example, if the cake is taken to be uniformly distributed on  $[0, 30]$ , then  $\alpha \approx 1.421358$ .  $\square$

As can be seen in the figure, the more risk-averse player 2 gets more than half when the cake is small and less than half when the cake is large.

**Proposition 3.2.** *Suppose that one party is risk-neutral, the other is risk-averse, and that the hypotheses of theorem 3.1 are satisfied. Then the amount of cake allocated to the risk-averse individual will be some fixed amount regardless of the outcome.*

**Figure 1**



This is effectively an insurance contract between the risk-averse and the risk-neutral party. This result should not come as a surprise and its appearance is a direct consequence of the Pareto Efficiency axiom (PE). The work of Borch (1962) on risk sharing informs us that all Pareto-efficient risk-sharing allocations, such as this bargaining one, will have this property.

We can do more to relate the Nash bargaining solution to the work of Borch. In the two-person case, Borch's celebrated result states that to each efficient allocation of risk there corresponds two positive real numbers  $\lambda_1$  and  $\lambda_2$  such that the efficient allocation maximises the function defined by:

$$\lambda_1 E(u_1(w(x))) + \lambda_2 E(u_2(x - w(x))).$$

The two scalars  $\lambda_1$  and  $\lambda_2$  trace out the Pareto frontier for the problem; it is routine to prove the following.

**Proposition 3.3.** *For the Nash bargaining solution, the Borch constants  $\lambda_1$  and  $\lambda_2$  are equal to  $E(u_1(\tilde{w}(y)))$  and  $E(u_2(y - \tilde{w}(y)))$  respectively, where  $\tilde{w}$  is a contract that maximises the Nash product.*

The next result focuses on the bargaining aspect of the problem.

**Proposition 3.4.** *If the risk aversion of the risk-averse party in proposition 3.2 increases, then the expected amount of cake he receives decreases.<sup>2</sup>*

The following conjecture generalises proposition 3.4.

**Conjecture 3.1.** *If the risk aversion of a player increases, then their expected share of the cake decreases.*

Here, we do not assume that the second party is risk-neutral. The result seems plausible but a proof remains elusive. Note that by “expected share of the cake”, we refer to the expectation of the contract rather than the contract itself. In particular, if the risk aversion of a party increases, this does *not* necessarily imply that their share of the cake becomes smaller everywhere.

### 3.1 Differing Beliefs

In order to reason about players with different beliefs, we need to rule out a potential pathology. Let  $F_i$  be the probability distribution associated with player  $i$ 's beliefs about the cake and let  $\mathfrak{B}$  denote the event space. We assume that

$$\forall_{(B \in \mathfrak{B})} (F_i(B) = 0 \iff F_j(B) = 0). \quad (4)$$

In words, if one player believes that an event will occur with probability zero, then so too does the other. This assumption rules out the degenerate case where one player believes an event will occur with positive probability while the other believes it will occur with zero probability. Intuitively, if this were to happen, then axiom (PE) would dictate that all the cake be given to the first player in that event. We choose to ignore uninteresting degenerate cases like this and focus instead on the case where players assign nonzero probabilities to the same events.

Note that (4) is not critical to our analysis; we can discard it simply by invoking the Lebesgue decomposition theorem. This theorem allows us to decompose  $F_i$  into  $F_i^1$  and  $F_i^2$ , where  $F_i^1$  is absolutely continuous with respect to  $F_j$  and  $F_i^2$  is singular with respect to  $F_j$ . We can then carry out our

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<sup>2</sup> We say that the Bernoulli utility function  $v$  is more risk averse than  $u$  if there exists a concave increasing function  $\phi$  such that  $u(\cdot) = \phi(v(\cdot))$ .

analysis on each part separately. This exercise would complicate the analysis but yield no new insights.

Before we begin, we introduce a useful concept from measure theory. The *Radon-Nikodym derivative* of  $F_2$  with respect to  $F_1$  is a measurable function  $g$  that satisfies

$$F_2(A) = \int_A g \, dF_1 \quad (5)$$

for every event  $A$ . Some authors write  $g$  as  $dF_2/dF_1$ , though one should be aware that, in general, this is not a conventional derivative. The existence of the Radon-Nikodym derivative is guaranteed by the Radon-Nikodym theorem. In the case where  $F_1$  and  $F_2$  are distributions of continuous random variables, the Radon-Nikodym derivative of  $F_2$  with respect to  $F_1$  is just the ratio of their density functions (where one density function is taken to be nonzero in the relevant domain). This can be verified by (5):

$$F_2(A) = \int_A f_2(t) \, dF_1 = \int_A \frac{f_2(t)}{f_1(t)} f_1(t) \, dt = \int_A \frac{f_2(t)}{f_1(t)} \, dF_1.$$

**Example 3.2.** Let  $\Omega = \{1, 2, 3\}$  and consider the following probability mass functions

$$p_1(x) = \begin{cases} 1/3 & : x = 1 \\ 1/3 & : x = 2 \\ 1/3 & : x = 3 \end{cases}, \quad p_2(x) = \begin{cases} 1/2 & : x = 1 \\ 1/4 & : x = 2 \\ 1/4 & : x = 3 \end{cases}.$$

Let  $P_1$  and  $P_2$  denote the distributions associated with these mass functions. Then the Radon-Nikodym derivative  $g$  of  $P_1$  with respect to  $P_2$  is given by

$$g(x) = \begin{cases} 2/3 & : x = 1 \\ 4/3 & : x = 2 \\ 4/3 & : x = 3 \end{cases}.$$

**Theorem 3.5.** *Consider a 2-person Nash bargaining problem where a random variable  $X$  is being shared. Suppose that the probability measures  $F_1$  and  $F_2$  denote the beliefs of the players, and that (4) holds. Let*

$$\begin{aligned} C_1 &= E[u_1(\tilde{w}(y))] = \int_{\Omega} u_1(\tilde{w}(y)) \, dF_1(y) \\ C_2 &= E[u_2(y - \tilde{w}(y))] = \int_{\Omega} u_2(y - \tilde{w}(y)) \, dF_2(y). \end{aligned}$$

*Then the amount  $\tilde{w} : \Omega \rightarrow \mathbb{R}$  given to player 1 must satisfy*

$$\frac{u'_1(\tilde{w}(x))}{u'_2(x - \tilde{w}(x))} = \frac{1}{g(x)} \frac{C_1}{C_2} \quad (F_1 - \text{almost everywhere}), \quad (6)$$

where  $g$  is the Radon-Nikodym derivative of  $F_2$  with respect to  $F_1$ .

In the case where both players believe that the cake is a continuous random variable with density function  $f_i$ , condition (6) can be written as

$$\frac{u'_1(\tilde{w}(x))}{u'_2(x - \tilde{w}(x))} = \frac{C_1 f_2(x)}{C_2 f_1(x)} \quad (\text{almost everywhere}). \quad (7)$$

The introduction of differing beliefs adds a great deal of realism to the problem. We see that beliefs can offset risk aversion — that is, an optimistic risk-averse player can take on more risk than a pessimistic but less risk-averse player.

No longer is there necessarily an “insurance” contract between risk-averse and risk-neutral players. We can still however say the following:

**Proposition 3.6.** *Given the hypotheses of theorem 3.5, suppose that one party is risk-averse and the other is risk-neutral. If the risk aversion of the risk-averse party increases, then the expected amount of cake he receives decreases.*

This proposition directly generalises proposition 3.4 to the case where the players have different beliefs.

## 3.2 $n$ -players

Finally, we generalise our result to the  $n$ -person case. The result, though a direct extension, is much more cumbersome to write.

**Theorem 3.7.** *Consider an  $n$ -person Nash bargaining problem where a random variable  $X$  is being shared. Suppose that the probability measure  $F_i$  denotes player  $i$ 's beliefs, and that (4) holds. Let  $(\tilde{w}_1(x), \dots, \tilde{w}_n(x))$  denote the amount of cake allocated to players 1 through  $n$  for cake size  $x$ . Let*

$$C_i = E[u_i(\tilde{w}_i(y))] = \int_{\Omega} u_i(\tilde{w}_i(y)) dF_i(y)$$

. Then  $(\tilde{w}_1(x), \dots, \tilde{w}_n(x))$  must satisfy

$$\frac{u'_i(\tilde{w}_1(x))}{u'_n(\tilde{w}_n(x))} = \frac{1}{g_i(x)} \frac{C_i}{C_n} \quad (F_i - \text{almost everywhere}) \quad (8)$$

for every  $i = 1, \dots, n - 1$ , where  $g_i$  is the Radon-Nikodym derivative of  $F_n$  with respect to  $F_i$ . We also have that  $\tilde{w}_n(x) \equiv x - \sum_{i=1}^{n-1} \tilde{w}_i(x)$ .

In the case where all players believe the cake is a continuous random variable, we can rewrite (8) as

$$\frac{u'_i(\tilde{w}_i(x))}{u'_n(\tilde{w}_n(x))} = \frac{C_i f_n(x)}{C_n f_i(x)} \quad (\text{almost everywhere}),$$

where  $f_i$  is the density function characterizing player  $i$ 's beliefs.

As expected, (8) gives  $n - 1$  first-order conditions.

## 4 Non-symmetric game

The axioms of a Nash bargaining game can be challenged. In particular, the Pareto-efficiency axiom (PE) can be questioned on the grounds that individual rationality need not imply collective efficiency – there are many strategic situations in which the Nash equilibrium is Pareto-inefficient. So why can we rule out Pareto-inefficient outcomes a priori? Furthermore, the axiom of symmetry can be questioned on the grounds that factors outside the model (factors beyond preferences and disagreement point) can easily result in asymmetric allocations. Both of these qualms can be dealt with relatively easily by replacing (PE) and (SYM) with the following:

**(SIR)** The solution  $c(S, d)$  strictly dominates the disagreement point  $d$ .

This axiom of “Strong Individual Rationality” is much easier to swallow. With this axiom in hand, it can be shown that a Bargaining solution must maximise a weighted version of the Nash product (see Roth 1979). In other words,

$$c(S, d) \in \operatorname{argmax} \mathbf{V} \in S \prod_{i=1}^n (V_i - d_i)^{\alpha_i},$$

where  $\alpha_i > 0$  and  $\sum \alpha_i = 1$ . This game has the added advantage of having a compelling noncooperative implementation (Rubinstein 1982). For the purposes of our analysis, not much changes in this non-symmetric version of

the Nash bargaining game. The proofs go through almost unchanged, with the exception that, instead of an equation like (7), we now have

$$\frac{u'_1(\tilde{w}(x))}{u'_2(x - \tilde{w}(x))} = \frac{\alpha_2 C_1 f_2(x)}{\alpha_1 C_2 f_1(x)} \quad (\text{almost everywhere}).$$

Since the only change to the first order equation is multiplication of the right hand side by a scalar, our results can be extended to the non-symmetric case without problems.

## Appendix: Proofs

Before we begin, we need some ancillary results from nonlinear analysis. For a more detailed treatment of this, consult a specialised text like Zeidler (1995, Chap 4.). If you are familiar with the basic properties of Gâteaux derivatives, the following subsection can be skipped.

### Mathematical machinery

**Definition 4.1.** Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed linear spaces, and  $T$  a subset of  $X$ . For any  $x_0$  in the interior of  $T$ , a map  $f : T \rightarrow Y$  is said to be **Gâteaux differentiable** at  $x_0$  if there is a bounded linear operator  $\delta_{f,x_0}$  such that

$$\lim_{t \rightarrow 0} \frac{\|f(x_0 + th) - f(x_0) - \delta_{f,x_0}(th)\|_Y}{t} = 0 \quad (9)$$

for all  $h \in X$ . We call the linear operator  $\delta_{f,x_0}$  the **Gâteaux derivative** of  $f$  at  $x_0$ .

**Proposition 4.1.** *Let  $S$  be an open subset of a normed linear space  $X$ , and  $f$  be a mapping from  $S$  into  $\mathbb{R}$ . Suppose that  $f$  is Gâteaux differentiable at the point  $x \in S$ . Then the functional*

$$\left. \frac{d}{dt} f(x + th) \right|_{t=0}$$

*is the Gâteaux derivative of  $f$  at  $x$  if and only if it is a linear functional of  $h$ .*

**Theorem 4.2** (Fermat's Theorem). *A necessary condition for a Gâteaux differentiable functional  $\Phi$  to have an extremum at  $\hat{x}$  is that the Gâteaux derivative  $\delta_{\Phi, \hat{x}}$  be the zero operator.*

## Proof of claims

With the aid of the above results, in particular Fermat's theorem (theorem 4.2), we can set to work proving the claims made in this paper.

*Proof of theorem 3.7.* Let  $w = (w_1, \dots, w_n)$  be each player's share of the cake. Since the shares must add to the total amount of cake available, we can write

$$w_n(x) \equiv x - \sum_{i=1}^{n-1} w_i(x).$$

By translating utility functions and using theorem 2.1, we simply need to maximise the Nash product

$$J(w) := \prod_{i=1}^n \mathbb{E}(u_i(w_i(x))),$$

to prove the theorem.

Let  $h = (h_1, \dots, h_{n-1}, 0)$ , where each  $h_i$  is a real-valued function of  $x$ . By proposition 4.1 and the product rule, the Gâteaux derivative of  $J$  is given by:

$$\begin{aligned} \delta_{J,w}(h) &= \left. \frac{d}{dt} J(w + th) \right|_{t=0} \\ &= \left. \frac{d}{dt} \left( \mathbb{E}(u_n(w_n - t \sum_{i=1}^{n-1} h_i)) \prod_{i=1}^{n-1} \mathbb{E}(u_i(w_i + th_i)) \right) \right|_{t=0} \\ &= \sum_{i=1}^{n-1} \left( \frac{d}{dt} \int_{\Omega} u_i(w_i + th_i) dF_i(x) \left( \prod_{j \neq i}^{n-1} \mathbb{E}(u_j(w_j + th_j)) \right) \mathbb{E}(u_n(w_n(x) - t \sum_{i=1}^{n-1} h_i)) \right) \\ &\quad + \left. \frac{d}{dt} \left( \int_{\Omega} u_n(w_n - t \sum_{i=1}^{n-1} h_i) dF_n(x) \right) \prod_{j \neq i}^{n-1} \mathbb{E}(u_j(w_j)) \right|_{t=0}, \end{aligned}$$

By invoking Leibniz's rule for Lebesgue integrals and evaluating at  $t = 0$ , we can simplify the above to

$$\begin{aligned} \delta_{J,w}(h) = & \sum_{i=1}^{n-1} \left( \mathbb{E}(u_n(w_n)) \prod_{j \neq i}^{n-1} \mathbb{E}(u_j(w_j)) \int_{\Omega} [u'_i(w_i(x)) h_i(x) dF_i(x)] \right) \\ & - \prod_{j=1}^{n-1} \mathbb{E}(u_j(w_j)) \int_{\Omega} \left[ u'_n(w_n(x)) \sum_{i=1}^{n-1} h_i(x) \right] dF_n(x), \end{aligned}$$

where  $u'_i$  denotes the derivative of  $u_i$ . By theorem 4.2, the Gâteaux derivative of  $J$  at the optimum contract  $\tilde{w}$  is identically zero. In particular,  $\delta_{J,\tilde{w}} = 0$  when  $h_k = 0$  for every  $j \neq i$ . Using this fact, we get  $n - 1$  first order conditions:

$$\begin{aligned} & \prod_{j \neq i}^n \mathbb{E}(u_j(\tilde{w}_j)) \int_{\Omega} u'_i(\tilde{w}_i(x)) h_i(x) dF_i(x) - \\ & \prod_{j=1}^{n-1} \mathbb{E}(u_j(\tilde{w}_j)) \int_{\Omega} u'_n(x - \sum_{i=1}^{n-1} \tilde{w}_i(x)) h_i(x) dF_n(x) = 0 \end{aligned}$$

for each  $i \in \{1, \dots, n-1\}$  and every admissible  $h_i$ . Since for nontrivial cakes  $E(u_k(\tilde{w})) \neq 0$  ( $1 \leq k \leq n$ ), we can divide through on both sides of the equation and write

$$\mathbb{E}(u_n(\tilde{w}_n)) \int_{\Omega} u'_i(\tilde{w}_i(x)) h_i(x) dF_i(x) - \mathbb{E}(u_i(\tilde{w}_i)) \int_{\Omega} u'_n(x - \sum_{i=1}^{n-1} \tilde{w}_i(x)) h_i(x) dF_n(x) = 0, \quad (10)$$

for every admissible  $h_i$ . By (4) and the Radon-Nikodym theorem, there exists a measurable function  $g_i$ , known as the Radon-Nikodym derivative of  $F_n$  with respect to  $F_i$ , that allows us to write (10) as

$$0 = \int_{\Omega} [\mathbb{E}(u_n(\tilde{w}_n)) u'_i(w(x)) - \mathbb{E}(u_i(\tilde{w}_i)) u'_n(\tilde{w}_n(x)) g_i(x)] h_i(x) dF_i(x).$$

By the fundamental lemma of the calculus of variations, we infer that

$$\mathbb{E}(u_n(\tilde{w}_n)) u'_i(\tilde{w}_i(x)) - \mathbb{E}(u_i(\tilde{w}_i)) u'_n(\tilde{w}_n(x)) g_i(x) = 0 \quad (F_i\text{-almost everywhere}).$$

Recall that the nonnegativity condition implies that  $\tilde{w}_n(x) \equiv x - \sum^{n-1} \tilde{w}_i(x)$ . So, by rearranging, we conclude that

$$\frac{\mathbb{E}(u_i(\tilde{w}_i(x)))}{\mathbb{E}(u_n(x - \sum^{n-1} \tilde{w}_j(x)))} = \frac{u'_i(w_i(x))}{u'_n(x - \sum^{n-1} \tilde{w}_j(x)) g_i(x)} \quad (F_i\text{-almost everywhere}).$$

This completes the proof. ■

*Proof of theorems 3.1 and 3.5.* These follow directly from theorem 3.7. ■

*Proof of proposition 3.2.* Without loss of generality, suppose that player 2 is risk-neutral. Then

$$u_2(x - w(x)) \equiv x - w(x).$$

Hence by theorem 3.1, we can write

$$u_1'(\tilde{w}(x)) = \frac{\mathbb{E}(u_1(\tilde{w}(x)))}{\mathbb{E}(x - \tilde{w}(x))}.$$

Since player 1 is risk-averse,  $u_1'$  is monotonic and therefore invertible. Thus we can write

$$\tilde{w}(x) = (u_1')^{-1}(\alpha) \quad (F - \text{almost everywhere})$$

where  $\alpha$  is the constant defined by:

$$\alpha := \frac{\mathbb{E}(u_1(\tilde{w}(x)))}{\mathbb{E}(x - \tilde{w}(x))}.$$

■

*Proof of proposition 3.4.* Suppose that the utility function of the risk-averse player is  $u_1$  and the contract is the constant  $w_1$ . Now suppose that  $u_1$  is replaced by  $u_2$ , where  $u_2$  is more risk-averse than  $u_1$  — that is  $u_2 \equiv \psi(u_1)$  for some increasing concave function  $\psi$ . Let  $w_2$  denote the contract corresponding to this Nash bargaining solution. It suffices to prove that  $w_1 > w_2$ .

Define a new utility function

$$v(t) := t\psi(u_1) + (1 - t)u_1,$$

and note that by theorem 4.2, the contract  $w$  for this situation solves

$$\frac{d}{dw}v(w) = \frac{v(w)}{\mathbb{E}(x) - w}.$$

Recall that by proposition 3.2,  $w$  is a constant in  $x$ . So, we can think of the optimum contract  $w$  as a real-valued function of  $t$ . It is easy to show that

this function is well-defined. We can differentiate  $w$  implicitly with respect to  $t$  to get

$$\begin{aligned} & \psi'(u_1(w))u_1'(w) + t\psi''(u_1(w))(u_1'(w))^2 \frac{dw}{dt} + \\ & t\psi'(u_1(w))u_1''(w) \frac{dw}{dt} + (1-t)u_1''(w) \frac{dw}{dt} - u_1'(w) \\ & = \frac{\psi'(u_1(w))u_1'(w) \frac{dw}{dt} (\mathbb{E}(x) - w) + \psi(u_1(w)) \frac{dw}{dt}}{(\mathbb{E}(x) - w)^2}. \end{aligned}$$

Rearranging this we get

$$\begin{aligned} & \frac{dw}{dt} \left[ t\psi''(u_1(w))(u_1'(w))^2 + t\psi'(u_1(w))u_1''(w) \right. \\ & \left. + (1-t)u_1''(w) - \frac{\psi'(u_1(w))u_1'(w)}{\mathbb{E}(x) - w} - \frac{\psi(u_1(w))}{(\mathbb{E}(x) - w)^2} \right] \\ & = (1 - \psi'(u_1(w)))u_1'(w). \end{aligned}$$

We know that the set

$$\{u_i(w(t)) : t \in [0, 1]\}$$

is a compact subset of strictly positive real numbers, since  $w(t)$  must maximise a Nash product. Hence this set contains a nonzero minimum  $\alpha$ . The number  $\psi'(\alpha) \geq \psi'(u_i(w(t))) > 0$  for  $t \in [0, 1]$ . Hence, by using axiom (AM) and dividing  $\psi$  by  $\psi'(\alpha)$ , we can assume without loss of generality that  $\psi'(w(t)) < 1$  for  $t \in [0, 1]$ . From this, we can unambiguously infer that

$$\frac{dw}{dt} < 0.$$

Therefore, we conclude that  $w(0) < w(1)$  — that is,  $w_1$  is greater than  $w_2$ . ■

*Proof of proposition 3.6.* The proof is similar to that of proposition 3.4. ■

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